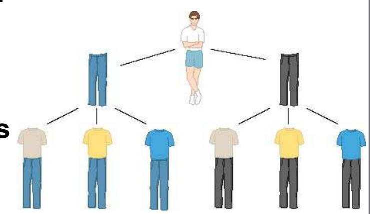
7.1-2: Probability Theory

1. Apply the Fundamental Counting Principle to determine number of different outcomes
2. Use the Fundamental Counting Principle to count permutations
3. Evaluate factorial expressions
4. Use the permutations formula
5. Use the combinations formula
6. Distinguish between permutation and combination problems
7. Compute theoretical probability.
8. Compute empirical probability.

7.1: Fundamental Counting Principle

❖ Definition
- ◆ If you can choose one item from a group of **M** items and a second item from a group of **N** items, then the total number of two-item choices is **M·N**.

❖ Tree Diagram
- ◆ A representation of all possible choices
- ◆ This tree diagram shows that there are 2·3 = 6 different outfits from 2 pairs of jeans and three T-shirts.



Applying the Fundamental Counting Principle

The Greasy Spoon Restaurant offers 6 appetizers and 14 main courses. In how many ways can a person order a two-course meal?

Solution:

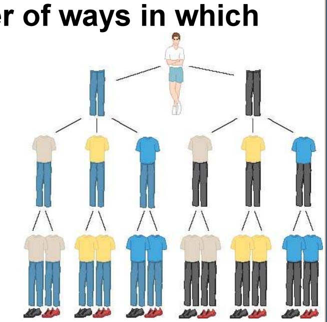
Choosing from one of 6 appetizers and one of 14 main courses, the total number of two-course meals is:

$$6 \cdot 14 = 84$$

Fundamental Counting Principle > Two Groups

❖ Definition
- ◆ The number of ways in which a series of successive things can occur is found by multiplying the number of ways in which each thing can occur
- ◆ The number of possible outfits from 2 pairs of jeans, 3 T-shirts, and 2 pairs of sneakers are:

$$2 \cdot 3 \cdot 2 = 12$$



Options in Planning a Course Schedule

Next semester, you are planning to take three courses: math, English and humanities.

There are 8 sections of math, 5 of English, and 4 of humanities that you find suitable. Assuming no scheduling conflicts, how many different three-course schedules are possible?

Solution:

This situation involves making choices with three groups of items.

Math	English	Humanities
{8 choices}	{5 choices}	{4 choices}

There are $8 \cdot 5 \cdot 4 = 160$ different three-course schedules.

Multiple-Choice Test – Answer Possibilities

You are taking a multiple-choice test that has ten questions.

Each of the questions has four answer choices, with one correct answer per question. If you select one of these four choices for each question and leave nothing blank, in how many ways can you answer the questions?

Solution:

This situation involves making choices with ten questions:

Question 1	Question 2	Question 3 · · ·	Question 9	Question 10
{4 choices}	{4 choices}	{4 choices}...	{4 choices}	{4 choices}

The number of different ways you can answer the questions is:

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^{10} = 1{,}048{,}576$$

Telephone Numbers in the United States

Telephone numbers in the United States begin with three-digit area codes followed by seven-digit local telephone numbers. Area codes and local telephone numbers cannot begin with 0 or 1.

How many different telephone numbers are possible?

Solution:

This situation involves making choices with ten groups of items. Here are the choices for each of the ten groups of items:

Area Code	Local Telephone Number
8 10 10	8 10 10 10 10 10 10

The total number of different telephone numbers is:

$$8 \cdot 10 \cdot 10 \cdot 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$
$$= 6{,}400{,}000{,}000$$

Permutations

- **Permutation** is an ordered arrangement of items that occurs when:
  1) The items are selected from the same group
  2) No item is used more than once
  3) The order of arrangement makes a difference
- **Example:** You need to arrange seven of your favorite books along a small shelf. How many different ways can you arrange the books, assuming that the order of the books makes a difference to you?

  Solution:

  You can choose any one of the seven books for the first position on the shelf. This leave six choices for the second position. After the first two positions are filled, there are five books to choose and so on.

  $$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

Factorial Notation

- The product 7·6·5·4·3·2·1 is called 7 factorial and is written 7! = 5040
- Definition:

  If *n* is a positive integer, the notation *n*! (read "*n* factorial") is the product of all positive integers from n down through 1.

  $n! = n(n-1)(n-2)\cdots(3)(2)(1)$

  0! (zero factorial), by definition, is 1.

  $0! = 1$

A Formula for Permutations

- Permutation is an ordered arrangement of items that occurs when:
  1) The items are selected from the same group
  2) No item is used more than once
  3) The order of arrangement makes a difference
- The number of possible permutations of *k* items are taken from *n* items is:

$${}_nP_k = \frac{n!}{(n-k)!}$$

a. $\frac{8!}{5!} = \frac{8\cdot7\cdot6\cdot\cancel{5!}}{\cancel{5!}} = 8\cdot7\cdot6 = 336$

b. $\frac{26!}{21!} = \frac{26\cdot25\cdot24\cdot23\cdot22\cdot\cancel{21!}}{\cancel{21!}} = 26\cdot25\cdot24\cdot23\cdot22 = 7{,}893{,}600$

Using the Formula for Permutations

- You and 19 of your friends have decided to form a business. The group needs to choose three officers– a CEO, an operating manager, and a treasurer. In how many ways can those offices be filled?

Solution:

Your group is choosing r = 3 officers from a group of n = 20 people. The order matters because each officer has different responsibilities:

$${}_nP_k = {}_{20}P_3 = \frac{20!}{(20-3)!} = \frac{20!}{17!} = \frac{20\cdot19\cdot18\cdot\cancel{17!}}{\cancel{17!}} = 20\cdot19\cdot18 = 6840$$

Combinations

- A combination of items occurs when:
  1) The items are selected from the same group
  2) No item is used more than once
  3) The order of items makes no difference
- Formula for Combinations
  - The number of possible combinations if *k* items are taken from *n* items is:

$${}_nC_k = \frac{n!}{(n-k)!k!}$$

- How many 3-person committees could be formed from 8 people?

Solution: We are selecting 3 people (*r* = 3) from 8 people (*n* = 8)

$${}_nC_k = {}_8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{\cancel{5!}3!} = \frac{8\cdot7\cdot6\cdot\cancel{5!}}{\cancel{5!}\cdot3\cdot2\cdot1} = 56$$

Is it a Permutation or Combination?

- Permutation problems involve situations in which order matters
- Combination problems involve situations in which the order of items makes no difference
- Determine which involve permutations and which involve combinations
  - Six people are on the board of supervisors for your neighborhood park. A three-person committee is needed to study the possibility of expanding the park. How many different committees could be formed from the six people? $_6C_3$
  - Six students are running for student government president, vice-president and treasurer. The student with the greatest number of votes becomes the president, the second highest vote-getter becomes vice-president, and the student who gets the third largest number of votes will be treasurer. How many different outcomes are possible? $_6P_3$

Combinations & Fundamental Counting Principle

- In December, 2009, the U.S Senate consisted of 60 Democrats and 40 Republicans. How many committees can be formed if each committee must have 3 Democrats and 2 Republicans?

Solution: **The order in which members are selected does not matter so this is a problem of combinations. Picking 3 Democrats out of 60.**

$$_{60}C_3 = \frac{60!}{(60-3)!3!} = \frac{60!}{57!3!} = \frac{60\cdot 59\cdot 58\cdot \cancel{57}!}{\cancel{57}!\cdot 3\cdot 2\cdot 1} = \frac{60\cdot 59\cdot 58}{3\cdot 2\cdot 1} = 34,220$$

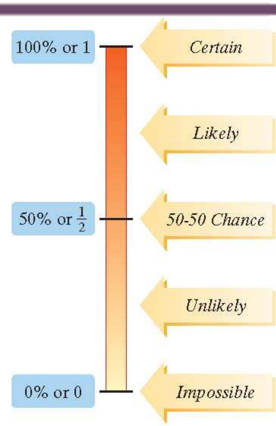
Select 2 Republicans out of 40.

$$_{40}C_2 = \frac{40!}{(40-2)!2!} = \frac{40!}{38!2!} = \frac{40\cdot 39\cdot \cancel{38}!}{\cancel{38}!\cdot 2\cdot 1} = \frac{40\cdot 39}{2\cdot 1} = 780$$

Use the Fundamental Counting Principle to find the number of committees that can be formed. $_{60}C_3 \cdot {_{40}C_2} = 34,220\cdot 780 = 26,691,600$

7.2: Probability

- ***Probabilities*** are assigned values from 0 to 1
  - The closer the probability of a given event is to 1, the more likely it is that the event will occur
  - The closer the probability of a given event is to 0, the less likely that the event will occur
- Express as Fractions or
- Express as Percentages or
- Express as Decimals or



Possible Values for Probabilities

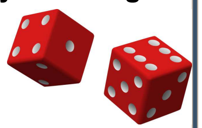
Theoretical Probability

- ***Experiment*** is any occurrence for which the outcome is uncertain
  - ***Sample space*** is the set of all possible outcomes of an experiment, denoted by ***S***
  - ***Event***, denoted by ***E*** is any subset of a sample space
  - Sum of the theoretical probabilities of all possible outcomes is 1
- If an event E has n(E) equally likely outcomes and its sample space S has n(S) equally-likely outcomes, the theoretical probability of event E, denoted by P(E), is:

$$P(E) = \frac{\text{number of outcomes in event E}}{\text{total number of possible outcomes}} = \frac{n(E)}{n(S)}$$

Computing Theoretical Probability

A die is rolled once. Find the probability of rolling:
a. 3 b. an even number



Solution:

The sample space is $S = \{1,2,3,4,5,6\}$

a. There is only one way to roll a 3 so $n(E) = 1$

$$P(3) = \frac{\text{number of outcomes that result in 3}}{\text{total number of possible outcomes}} = \frac{n(1)}{n(6)} = \frac{1}{6}$$

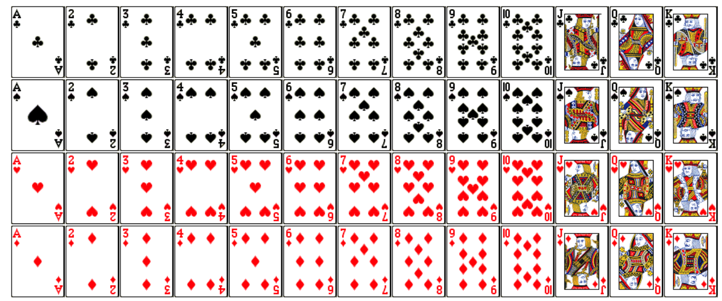
b. Rolling an even number describes the events $E = \{2,4,6\}$
This event can occur in 3 ways: $n(E) = 3$.

$$P(\text{even}) = \frac{\text{number of outcomes that result in even}}{\text{total number of possible outcomes}} = \frac{n(3)}{n(6)} = \frac{3}{6} = \frac{1}{2}$$

 17

Probability and a Deck of 52 Cards

You are dealt one card from a standard 52-card deck. Find the probability of being dealt a
a. king b. a heart c. one eyed jack



 18

Empirical Probability

Applies to situations in which we <u>observe</u> how frequently an event occurs. The empirical probability of event E is:

$$P(E) = \frac{\text{Observed number of times Event occurs}}{\text{Total number of observed occurances}} = \frac{n(E)}{n(S)}$$

	Never Married	Married	Widowed	Divorced	Total
Male	37.5	64.7	2.7	9.6	114.5
Female	31.7	65.2	11.2	13.2	121.3
Total	69.2	129.9	13.9	22.8	235.8

If one person is randomly selected from the population described above, find the probability that the person is female.

$$P(female) = \frac{\text{females}}{\text{total number of adults}} = \frac{121.3}{235.8} \approx 0.51$$

 19

Winning the Lottery

Florida's lottery game, LOTTO, is set up so that each player chooses six different numbers from 1 to 53. With one LOTTO ticket, what is the probability of winning this prize?

Solution:

Because the order of the six numbers does not matter, this situation involves combinations:

$$P(\text{winning}) = \frac{\text{number of ways of winning}}{\text{total number of possible combinations}}$$

$$_{53}C_6 = \frac{53!}{(53-6)!6!} = \frac{53!}{47!6!} = \frac{53\cdot52\cdot51\cdot50\cdot49\cdot48\cdot\cancel{47!}}{\cancel{47!}\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = 22{,}957{,}480$$

$$P(\text{winning}) = \frac{1}{22{,}957{,}480} \approx 0.0000000436$$

 20