# 6

Students learn about finance as it applies to their daily lives. Two of the most important types of financial decisions for many people involve either buying a house or saving for retirement. Students explore retirement accounts and mortgages. Understanding the quantitative side helps people make better decisions.

Course Outcomes:

- Recognize and apply mathematical concepts to real-world situations
- Efficiently use relevant technology
- Identify and solve problems in finance

6.1 Simple Interest

Simple interest concepts are developed. Students learn to find simple interest and future value. Simple interest rate and principle are found using the future value formula and algebra.

6.2 Compound Interest

Compound interest concepts are developed. Students solve compound interest problems using their calculators and compound interest formulas for present value, future value, and future value for continuous compounding.

6.3 Annuities

Students learn to find periodic payments and future value for annuities. There is a focus on retirement accounts. The advantage of time is explored.

6.4 Mortgages

The different aspects of mortgages are discussed: down payment, points, monthly payments, interest. Students use the formula for monthly payment to better understand the advantages of shorter mortgages as opposed to longer mortgages.

People have been borrowing and lending money for millennia. Whether you borrow money to buy a car or put money in the bank, there will be some type of interest involved. The first type of interest that we will study is called simple interest. The formula for simple interest is

$\text{Int} = \text{Prt}$

$\text{Int} =$ simple interest (a dollar amount)
$\text{P} =$ principal or present value which is the amount borrowed
$\text{r} =$ annual simple interest rate
$\text{t} =$ time measured in years

For simple interest the amount is applied one time at the end of the borrowing period. Notice that when we talk about interest, we are referring to a dollar amount. When we talk about simple interest rate, we are talking about the percent of the principal.

Examples

3. A construction company borrows $30,000 for 2 years at 5% simple interest. How much interest must the construction company pay for the use of the money?

| Steps | Reasons |
|---|---|
| $\text{Int} = \text{Prt}$ | Recognize the type of interest and write down the formula. |
| $\text{Int} =$ ??? | We are looking for the interest, which will be a dollar amount. |
| $\text{P} =$ $30,000 | We have the amount borrowed, which is $\text{P}$. |
| $\text{r} =$ 5% or .05 | The simple interest rate is given. |
| $\text{t} =$ 2 years | The number of years that the money is borrowed. |
| $\text{Int} = \text{Prt}$ | Substitute the numbers into the formula. |
| $\text{I} = 30{,}000 \cdot .05 \cdot 2$ | Use a calculator to find the simple interest $\text{I}$ . |
| $\text{I} = \$3000$ | |

The construction company must pay $3000 in simple interest.

4. Find the simple interest for a principal of $5400 that is borrowed at simple interest rate 3% for a period of 8 months.

| Steps | Reasons |
|---|---|
| $\text{Int} = \text{Prt}$ | Recognize the type of interest and write down the formula. |
| $\text{Int} =$ ??? | We are looking for the interest, which will be a dollar amount. |
| $\text{P} =$ $5400 | The principal is $\text{P}$. |
| $\text{R} =$ 3% or .03 | The simple interest rate is given. |
| $\text{t} =$ 8 months | The number of years that the money is borrowed. |

Here we have to make an adjustment. The 8 months needs to be converted to years. Many students will quickly divide the 8 months by 12 to get the number of years, which is correct. There is a more organized way.

$$t = 8 \text{ months} \cdot \frac{1 \text{ year}}{12 \text{ months}} = \frac{8}{12} \text{ or } \frac{2}{3} \text{ years}$$

The fraction $\frac{1 \text{ year}}{12 \text{ months}}$ is called a unit fraction. It has a value of 1. So, we do not change the value of the time. It is true that $8 \text{ months} = \frac{2}{3} \text{ year}$. This organized method of changing the units is called dimensional analysis.

| | |
|---|---|
| $\text{Int} = \text{Prt}$ | Substitute the numbers into the formula. |
| $\text{Int} = 5400 \cdot .03 \cdot \left(\frac{2}{3}\right)$ | Use a calculator to find the simple interest $\text{Int}$ . |
| $\text{Int} = \$108$ | The simple interest is $108. |

When substituting for the number of years it is important to use parentheses. That way we can just type what we see into the calculator:
5400 multiplication .03 multiplication (2 divide 3) =

## Future Value

Rather than asking how much interest will be paid, we may want to know how much money will be paid at the end of the borrowing time (or investment period). This end amount is called the future value. To calculate the future value, we could just add the principal plus interest. There is also a future value formula for simple interest:

$\text{FV} = \text{P}\,(1 + \text{rt})$

$\text{FV}$ = future value for simple interest
$\text{P}$ = principal or present value which is the amount borrowed
$\text{r}$ = annual simple interest rate
$\text{t}$ = time measured in years

We can talk about the accumulated amount, total amount paid, and the amount at the end of the investment. All are examples of future value. The best way to think of present value or principal and future value is with a time line.

Present Value ⟶ time invested or borrowed ⟶ Future Value
Earlier time                                                  Later in time

Examples

3. If a principal of \$2800 is invested for a period of 5 years at a simple interest rate of 3.4%, what will be the future value?

| Steps | Reasons |
| --- | --- |
| $FV = P\,(1 + rt)$ | Recognize the type of interest and write future value for simple interest formula. |
| $FV = ???$ | We are looking for the future value. |
| $P = \$2800$ | We have the principal, $P$. |
| $r = 3.4\%$ or .034 | The simple interest rate is given. |
| $t = 5$ years | The number of years that the money is borrowed. |

$FV = P\,(1 + rt)$ Substitute the numbers into the formula.
$FV = 2800(1 + .034 \cdot 5)$
$FV = \$3276$ Use a calculator to find the future value FV .

Be sure to write down every symbol in the formula, replace the variables without losing any of the parentheses or operations, and then type every symbol into the calculator.

4. If \$12,000 is borrowed for 120 days at 5.2% simple interest, how much must be paid back at the end?

| Steps | Reasons |
| --- | --- |
| $FV = P\,(1 + rt)$ | We are asked for the end amount to be paid back at the end of the borrowing time. So, use the future value formula for simple interest. |
| $FV = ???$ | We are looking for the future value. |
| $P = \$12,000$ | The amount borrowed, $P$, comes before the time borrowed. |
| $r = 5.2\%$ or .052 | The simple interest rate is given. |
| $t = 120$ days | The time that the money is borrowed. |

Again we need to make an adjustment for the time. The 120 days needs to be converted to years. Often students will just divide the 120 days by 365 to get the number of years, which is correct. We can also use dimensional analysis.

$$t = 120 \text{ days} \cdot \frac{1 \text{ year}}{365 \text{ days}} = \frac{120}{365} \text{ years}$$

The fraction $\frac{1 \text{ year}}{365 \text{ days}}$ is the appropriate unit fraction because it has a value of 1. Sometimes 360 days is used in the denominator, which is a very close approximation.

| | |
|---|---|
| $\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$ | Substitute the numbers into the formula. |
| $\mathrm{FV} = 12{,}000\left(1 + .052 \cdot \left(\frac{120}{365}\right)\right)$ | Use a calculator to find the future value FV. |
| $\mathrm{FV} = \$12{,}205.15$ and $\$12{,}205$ | Round off to the nearest penny or dollar. |

It may be that we have all the information except the present value or simple interest rate or time. In those cases, once we have plugged the other values into the formula we can solve for the present value or simple interest rate or time algebraically.

Examples

5. If the future value is $632.40 and the present value is $600 for a 9 month investment, what is the simple interest rate?

| Steps | Reasons |
|---|---|
| $\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$ | We are asked about a problem involving future value for simple interest. |
| FV = 632.40 | The future value is given. |
| P = $600 | The present value is given. |
| r = ??? | Find the simple interest rate. |
| t = 9 months | The time that the money is borrowed. |

Use dimensional analysis to change months to years.

$$\mathrm{t} = 9 \text{ months} \cdot \frac{1 \text{ year}}{12 \text{ months}} = \frac{9}{12} \text{ or } \frac{3}{4} \text{ or } 0.75 \text{ years}$$

| | |
|---|---|
| $\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$ | Substitute the numbers into the formula. |
| $632.40 = 600(1 + \mathrm{r} \cdot 0.75)$<br>$632.4 = 600 + 450\mathrm{r}$<br>$632.4 - 600 = 450r$<br>$32.4 = 450\mathrm{r}$<br>$\frac{32.4}{450} = \mathrm{r}$ | Now solve the equation for r. Begin with the distributive property because of the + inside the parentheses. |
| $\mathrm{r} = .072 \text{ or } 7.2\%$ | Write the simple interest rate as a percent. |

6. How much money needs to be invested at 4.8% simple interest for three years to meet a future value goal of $17,000? Round-up to the nearest dollar.

| Steps | Reasons |
| --- | --- |
| $\text{FV} = \text{P}\,(1 + \text{rt})$ | We are asked about a problem involving future value for simple interest. |
| FV = $17,000 | The future value is given. |
| P = ??? | We are asked for the present value. |
| $\text{r} = 4.8\%$ or $.048$ | The simple interest rate is given. |
| t = 3 years | The time that the money is borrowed. |
| $\text{FV} = \text{P}\,(1 + \text{rt})$ | Substitute the numbers into the formula. |
| $17{,}000 = \text{P}(1 + .048 \cdot 3)$<br>$17{,}000 = \text{P}(1.144)$<br>$\frac{17{,}000}{1.144} = \text{P}$<br>$\text{P} = 14{,}860.139 \ldots$ | Now solve the equation for r. Begin with the distributive property because of the + inside the parentheses. |
| $\text{P} = \$14{,}861$ | |

We are in the habit of always rounding-up when finding the present value. We do this upwards rounding for present value because that will assure that the future value goal is met. Generally, it is not of huge importance, but financial advisors should want to make sure that the goal is met or surpassed. If we round-down even by a little bit, then the goal is not quite met.

As we go through the different types of formulas, we will need to determine which formula we will need to use. Here the formulas talk about simple interest. We need to determine whether we want the interest, the future value (end amount or accumulated value), or the present value (the amount before the time the money grows).

Exercises

Solve and round to the nearest penny:

1. Find the simple interest for a principal of $10,400 that is borrowed at simple interest rate 4% for a period of 5 years.

2. Find the simple interest for a principal of $3000 that is borrowed at simple interest rate 2.5% for a period of 3 years.

3. Find the simple interest for a principal of $120,000 that is borrowed at simple interest rate 12.5% for a period of 15 years.

4. Find the simple interest for a principal of $15,000 that is borrowed at simple interest rate 9.25% for a period of 10 years.

5. Find the simple interest for a principal of $5,400 that is borrowed at simple interest rate 4% for a period of 9 months.

6. Find the simple interest for a principal of $17,200 that is borrowed at simple interest rate 3.5% for a period of 8 months

7. Find the simple interest for a principal of $20,500 that is borrowed at simple interest rate 4.25% for a period of 8 months.

8. Find the simple interest for a principal of $41,000 that is borrowed at simple interest rate 12% for a period of 3 months.

9. Find the simple interest for a principal of $15,000 that is borrowed at simple interest rate 4.5% for a period of 100 days.

10. Find the simple interest for a principal of $8,700 that is borrowed at simple interest rate 2.8% for a period of 60 days.

11. Find the simple interest for a principal of $325,000 that is borrowed at simple interest rate 12.5% for a period of 150 days.

12. Find the simple interest for a principal of $200,000 that is borrowed at simple interest rate 7.8% for a period of 45 days.

13. If a principal of $20,000 is invested for a period of 8 years at a simple interest rate of 3.7%, what will be the future value?

14. If a principal of $14,000 is invested for a period of 10 years at a simple interest rate of 5.2%, what will be the future value?

15. If a principal of $15,000 is invested for a period of 8 months at a simple interest rate of 7.5%, what will be the future value?

16. If a principal of $200,000 is invested for a period of 4 months at a simple interest rate of 6.5%, what will be the future value?

17. If a principal of $14,600 is invested for a period of 90 days at a simple interest rate of 10.5%, what will be the future value?

18. If a principal of $28,000 is invested for a period of 30 days at a simple interest rate of 7.25%, what will be the future value?

Find the simple interest rate:

19. If the future value is $9560 and the present value is $8000 for a 3 year investment, what is the simple interest rate?

20. If the future value is $15,125 and the present value is $12,500 for a 5 year investment, what is the simple interest rate?

21. If the future value is $18,342 and the present value is $18,000 for a 3 month investment, what is the simple interest rate?

22. If the future value is $24,252 and the present value is $23,500 for a 6 month investment, what is the simple interest rate?

Find the principal (also called present value). Round-up to the nearest dollar

23. How much money needs to be invested at 3.4% simple interest for six years to meet a future value goal of $15,400? Round-up to the nearest dollar.

24. How much money needs to be invested at 8.25% simple interest for eight years to meet a future value goal of $120,000? Round-up to the nearest dollar.

25. How much money is borrowed at 12.4% simple interest for 90 days if the amount paid after the 90 days is $9,000? Round-up to the nearest dollar.

26. How much money is borrowed at 3.25% simple interest for 120 days if the amount paid after the 120 days is $15,000? Round-up to the nearest dollar.

Answer the following, and round to the nearest dollar. If asked to find the principal (or present value), round-up to the nearest dollar.

27. Find the interest paid on a $5000 simple interest loan for 45 days at 11.25%.

28. How much money was invested eight years ago at 4.25% simple interest if today's value is $17,500?

29. What is the amount due on a $25,000 loan for 120 days at simple interest rate 9.9%?

30. Find the interest paid on an 80 day $125,000 loan at 8.5% simple interest.

31. What is the simple interest rate if a $7000 investment for five years has a future value of $8300? (Round to the nearest tenth of a percent.)

32. What is the simple interest rate if a $2850 investment for two years has a future value of $3150? (Round to the nearest tenth of a percent.)

33. How much money was invested twenty years ago at 5.25% simple interest if today's value is $100,000?

34. What is the amount due on a $5,000 loan for 90 days at simple interest rate 12.2%?

With simple interest the interest is applied only once at the end of the time invested or borrowed. The more typical situation is compound interest where the interest is applied after a certain amount of time. Then that interest grows for the rest of time. For instance, most regular savings accounts have interest compounded daily. After depositing money into the account at the end of the first day, a little bit of interest is earned. That little bit of interest is then in the account and it also earns interest for the rest of the time that the money is in the account. At the end of the second day a little more interest is earned, which also grows for the rest of the time that the money is in the account. The process continues where the interest earned is put into the account daily to grow for the rest of the time.

We have a formula for the future value for compound interest:

$$\mathrm{FV} = \mathrm{P}\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$$

$\mathrm{FV}$ = future value for compound interest
$\mathrm{P}$ = principal or present value
$\mathrm{r}$ = annual compound interest rate
$\mathrm{t}$ = time measured in years
$\mathrm{n}$= number of compounding per year

From simple interest we know what most of the letters stand for, but we should take a closer look at the value of $\mathrm{n}$.

| | |
|---|---|
| Compounded daily | $\mathrm{n}$ = 365 or 360 to round-off |
| Compounded monthly | $\mathrm{n}$ = 12 |
| Compounded quarterly | $\mathrm{n}$ = 4 |
| Compounded semi-annually | $\mathrm{n}$ = 2 |
| Compounded annually | $\mathrm{n}$ = 1 |

Examples:

1. $15,000 is invested for six years at 4.7% interest compounded monthly. What is the future value?

| Steps | Reasons |
|---|---|
| $\mathrm{FV} = \mathrm{P}\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$ | Recognize the type of interest. We see the words compound interest and we are asked for future value. |
| $\mathrm{FV}$ = ??? | We are looking for the future value. |
| $\mathrm{P}$ = $15,000 | We have the principal, $\mathrm{P}$. |
| $\mathrm{r}$ = 4.7% or .047 | The compound interest rate is given. |
| $\mathrm{t}$ = 6 years | The number of years that the money is borrowed. |
| $\mathrm{n}$ = 12 | Compounded monthly means $\mathrm{n}$ = 12 for 12 times per year. |
| $\mathrm{FV} = \mathrm{P}\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$ | Substitute the numbers into the formula. |
| | Use a calculator to find the future value $\mathrm{FV}$ . |

$$\mathrm{FV} = 15{,}000\left(1 + \frac{.047}{12}\right)^{12\cdot 6}$$

The future value is $19,875.73. Round the future value to the nearest penny or dollar.

2. What is the accumulated value for a $50,000 loan for 3.5 years at 6% interest compounded daily?

| Steps | Reasons |
|---|---|
| $\mathrm{FV} = \mathrm{P}\left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$ | Recognize the type of interest. We see the words compound interest and we are asked for the accumulated value, which is the future value. |
| $\mathrm{FV}$ = ??? | We are looking for the future value. |
| $\mathrm{P}$ = $50,000 | We have the principal, $\mathrm{P}$. |
| $\mathrm{r}$ = 6% or .06 | The compound interest rate is given. |
| $\mathrm{t}$ = 3.5 years | The number of years that the money is borrowed. |
| $\mathrm{n}$ = 365 | Compounded daily means $\mathrm{n}$ = 365 for 365 times per year. Sometimes $\mathrm{n}$ = 360 is used. |
| $\mathrm{FV} = \mathrm{P}\left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$ | Substitute the numbers into the formula. |
| $\mathrm{FV} = 50{,}000\left(1 + \frac{.06}{365}\right)^{365\cdot 3.5}$ | Use a calculator to find the future value $\mathrm{FV}$ . |

The accumulated value is $61,682.84.

Less frequently we may see the future value for interest compounded continuously. Imagine that instead of daily we had the compounding period every hour or every minute or every second or every part of a second. Then we would be approaching continuous compounding. The formula for continuous compound interest is

$$\mathrm{FV} = \mathrm{P}e^{\mathrm{rt}}$$

$\mathrm{FV}$ = future value for continuous compound interest
$\mathrm{P}$ = principal or present value
$\mathrm{r}$ = annual continuous compound interest rate
$\mathrm{t}$ = time measured in years

The letter $e$ is actually a number. It is an irrational number like the number $\pi$. Since $e$ written as a decimal number goes on forever without repeating, we have to use a symbol to express the number exactly. The number $e$ is approximately equal to 2.71828…

Example:

3. Inflation is often calculated using continuous compound interest. If the inflation rate is 2.5% compounded continuously, how much will a $200 cart of groceries cost in 20 years?

| Steps | Reasons |
| --- | --- |
| $FV = Pe^{rt}$ | Recognize the type of interest. We see the words continuous compound interest and we are asked for cost in 20 years, which is the future value. |
| $FV = ???$ | We are looking for the future value. |
| $P = \$200$ | We have the principal or present value, $P$. |
| $r = 2.5\%$ or .025 | The continuous compound interest rate is given. |
| $t = 20$ years | The number of years. |
| $FV = Pe^{rt}$<br>$FV = 200e^{.025 \cdot 20}$ | Substitute the numbers into the formula. Use a scientific calculator to find the future value $A$. |

The cart of groceries will cost $329.74 in twenty years.

As with simple interest, we may have the future value and want to know the present value for compound interest. Rather than using the future value formula and solving for the present value like we did with simple interest, we will use a present value formula for compound interest, which is just an algebraic manipulation of the future value formula for compound interest that we have above.

Present value for compound interest:

$$P = \frac{FV}{\left(1 + \frac{r}{n}\right)^{nt}}$$

$FV$ = future value for compound interest
$P$ = principal or present value
$r$ = annual compound interest rate
$t$ = time measured in years
$n$= number of compounding per year

Examples:

4. A salesperson receives a large bonus for having the best sales record for the year. If she wants to have $25,000 in five years for the down payment on a house, how much must she put aside now at 7.5% compounded quarterly?

Steps | Reasons

$P = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$ Recognize the type of interest. We see the words compound interest and we are asked for the amount now before the investment. We are looking for the present value with compound interest.

$FV = \$25,000$ We have the future value $FV$.
$P = ???$ We are looking for the present value, $P$.
$r = 7.5\%$ or .075 The compound interest rate is given.
$t = 5$ years The number of years that the money is borrowed.
$n = 4$ Compounded quarterly means $n = 4$ for 4 times per year.

$P = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$ Substitute the numbers into the formula.

Use a calculator to find the present value $P$ .

$$P = \frac{25{,}000}{\left(1+\frac{.075}{4}\right)^{4\cdot 5}}$$

The salesperson needs to put aside $17,242.

For present value problems we should always round-up to the nearest dollar or nearest penny. That way we meet or exceed the goal. If rounding leads to rounding-down, then we just miss the stated goal.

5. A twenty-five year old believes he will need $750,000 to retire comfortably. How much will he need to put aside now at 3.25% interest compounded monthly to meet his goal in 40 years?

Steps | Reasons

$P = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$ Recognize the type of interest. We see the words compound interest and we are asked for the amount now before the investment. We are looking for the present value with compound interest.

$FV = \$750,000$ We have the future value goal, $FV$.
$P = ???$ We are looking for the present value, $P$.
$r = 3.25\%$ or .0325 The compound interest rate is given.
$t = 40$ years The number of years that the money is borrowed.
$n = 12$ Compounded quarterly means $n = 12$ for 12 times per year.

$P = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$ Substitute the numbers into the formula.

Use a calculator to find the present value, $P$ .

$$\mathrm{P} = \frac{750{,}000}{\left(1 + \frac{.0325}{12}\right)^{12\cdot 40}}$$

The person needs to put aside $204,758.34 in order to meet the retirement goal.

That is amazing! By putting aside $204,758.34 at 3.25% interest compounded monthly, a twenty-five year old can meet a retirement goal of $750,000. The other $545,241.66 all comes from interest earned on the investment. The interest is the future value minus the present value (750,000-204,758.34 = 545,241.66) because the present value is the amount put in to the account. So, why doesn't everybody just put aside about $200,000 when they are 25 years old? What do we do instead? In the next chapter we will learn about savings plans with periodic (monthly) payments called annuities.

## Exercises

For future value problems, round to the nearest dollar. For present value problems, round-up to the nearest dollar.

1. $18,000 is invested for ten years at 3.7% interest compounded monthly. What is the future value?

2. $75,000 is invested for four years at 1.5% interest compounded monthly. What is the future value?

3. What is the accumulated value for a $150,000 loan for twenty-five years at 5.25% interest compounded daily?

4. What is the accumulated value for an $80,000 loan for eight years at 6% interest compounded daily?

5. $150,000 is invested for seventeen years at 4.75% interest compounded semiannually. What is the future value?

6. $5,000 is invested for 5.5 years at 4.1% interest compounded semiannually. What is the future value?

7. What is the accumulated value for a $10,000 loan for twelve years at 8% interest compounded quarterly?

8. What is the accumulated value for a $500,000 loan for fifteen years at 5.9% interest compounded quarterly?

9. $30,000 is invested for twenty years at 11.5% interest compounded monthly. What is the future value?

10. $320,000 is invested for fifteen years at 1.5% interest compounded monthly. What is the future value?

11. $60,000 is invested for twenty years at 6.4% interest compounded annually. What is the future value? How is annual compounding different from simple interest?

12. What is the accumulated value for a $45,000 loan for fifteen years at 7.25% interest compounded annually? How is annual compounding different from simple interest?

13. What is the accumulated value for a $95,000 loan for 8.5 years at 3.25% interest compounded daily?

14. What is the accumulated value for a $190,000 loan for five years at 2% interest compounded daily?

15. $50,000 is invested for fourteen years at 2.75% interest compounded semiannually. What is the future value?

16. $40,000 is invested for 4.5 years at 1.75% interest compounded semiannually. What is the future value?

17. What is the accumulated value for an $800,000 loan for four years at 7.5% interest compounded quarterly?

18. What is the accumulated value for a $1,500,000 loan for twenty years at 6.8% interest compounded quarterly?

19. $32,000 is invested for ten years at 4.5% interest compounded continuously. What is the future value?

20. $102,000 is invested for twenty-two years at 3.5% interest compounded continuously. What is the future value?

21. What is the accumulated value for a $7,000 loan for four years at 5.5% interest compounded continuously?

22. What is the accumulated value for a $68,000 loan for twelve years at 3.75% interest compounded continuously?

23. Inflation is often calculated using continuous compound interest. The cost of a mixture of items and the salary that it takes to purchase these items grows continuously. If the inflation rate is 2.5% compounded continuously, how large of a salary will somebody need in thirty years to have the same buying power as a $30,000 salary in today's dollars?

24. Inflation is often calculated using continuous compound interest. The cost of a mixture of items and the salary that it takes to purchase these items grows continuously. If the inflation rate is 1.5% compounded continuously, how large of a salary will somebody need in forty years to have the same buying power as a $25,000 salary in today's dollars?

25. If the inflation rate is 2.25% compounded continuously and the price of gasoline follows this inflation rate, how much will a $50 tank of gasoline cost in ten years?

26. If the inflation rate is 1.85% compounded continuously and the price of groceries follows this inflation rate, how much will a $175 cart of groceries cost in twenty years?

27. An investment grows at 5% compounded daily for ten years. If the future value of the investment is $40,000, what is the present value?

28. An investment grows at 3.25% compounded daily for fifteen years. If the future value of the investment is $25,000, what is the present value?

29. An investment grows at 4.2% compounded quarterly for twelve years. If the future value of the investment is $100,000, what is the present value?

30. An investment grows at 7.5% compounded quarterly for eight years. If the future value of the investment is $100,000, what is the present value?

31. How much money must be invested today at 6.7% interest compounded monthly so that there is an accumulated value of $50,000 in five years?

32. How much money must be invested today at 3.4% interest compounded monthly so that there is an accumulated value of $40,000 in fifteen years?

33. How much money can be borrowed at 3.65% interest compounded semiannually, if the borrower is willing to pay back $75,000 in ten years?

34. How much money can be borrowed at 4.25% interest compounded semiannually, if the borrower is willing to pay back $15,000 in two years?

35. A salesperson receives a large bonus for having the best sales record for the year. If she wants to have $15,000 in three years for the down payment on a house, how much must she put aside now at 8.5% compounded quarterly?

36. A salesperson receives a large bonus because the company meets all of the objectives and she has skills that the company feels are irreplaceable. If she wants to have $40,000 in six years for the down payment on a house, how much must she put aside now at 9.5% compounded quarterly?

37. If today's value of a government savings bond is $20,000, how much was invested seven years ago? The interest rate was 6.5% compounded daily.

38. If today's value of a government savings bond is $3,000, how much was invested ten years ago? The interest rate was 3.5% compounded daily.

39. A forty year old believes he will need $1,000,000 to retire comfortably. How much will he need to put aside now at 4.25% interest compounded semiannually to meet his goal in 25 years?

40. A twenty year old believes he will need $2,000,000 to retire comfortably. How much will he need to put aside now at 3.25% interest compounded semiannually to meet his goal in 45 years?

41. A young person is trying to figure out how to retire comfortably, which she believes will take $1,500,000.
    a. How much must a twenty-five year old put aside for 40 years at 5% interest compounded monthly to reach the goal?
    b. How much must a forty-five year old put aside for 20 years at 5% interest compounded monthly to reach the goal?
    c. How much must the twenty-five year old put aside for 40 years at 7% compounded monthly to reach the goal? 3% compounded monthly?
    d. The twenty-five year old may have other financial obligations – family, housing, school loans to pay back. What else can be done aside from putting aside a huge amount of money all at once?

42. A young person is trying to figure out how to retire comfortably, which he believes will take $2,000,000.
    a. How much must a twenty year old put aside for 45 years at 4.3% interest compounded quarterly to reach the goal?
    b. How much must a forty-five year old put aside for 20 years at 4.3% interest compounded quarterly to reach the goal?
    c. How much must the twenty year old put aside for 45 years at 7.5% compounded quarterly to reach the goal? 2.8% compounded quarterly?

d. The twenty year old may have little money and other financial obligations – food, housing, credit card debt, living the good life. What else can be done aside from putting aside a huge amount of money all at once?

43. A teacher decides to put aside some money to have for a rainy day, which fortunately never seems to come. First he puts aside $10,000 for five years at 2.25% compounded monthly interest. After five years the economy changes and he is able to take that money and put it into an account that earns 6.4% interest compounded semiannually for twenty years. How much does the teacher have at the end of the twenty-five year investment time?

44. A doctor decides to put aside some money to have for a rainy day, which fortunately never seems to come. First she puts aside $25,000 for three years at 3.25% interest compounded monthly interest. After three years the economy changes and she is able to take that money and put it into an account that earns 6.4% interest compounded quarterly for fifteen years. How much does the doctor have at the end of the eighteen year investment time?

Often we do not have enough money to meet a future financial goal all at one time. An annuity is a fixed sum paid over equal periods per year over some number of years at a set interest rate. A very relevant example for most people is their retirement account. For instance, somebody saving for retirement may put aside $400 per month for thirty years. If the interest and payment schedule stay the same, we are talking about an annuity.

Formula for future value of an annuity:

$$\mathrm{FV} = \frac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$$

$\mathrm{FV} =$ future value for an annuity
$\mathrm{Pmt} =$ periodic payment
$\mathrm{r} =$ annual compound interest rate
$\mathrm{t} =$ time measured in years
$\mathrm{n} =$ number of payments, which is the same as the number of compounding per year

Examples:

1. If $400 is put aside every month at 5% interest compounded monthly for 30 years, what is the accumulated value? How much of that accumulated value is interest?

| Steps | Reasons |
|---|---|
| $\mathrm{FV} = \frac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$ | Since we are putting aside a set amount each month for thirty years at a constant interest rate, we have an annuity. |
| $\mathrm{FV} =$ ??? | We are looking for the accumulated value or future value of the annuity. |
| $\mathrm{Pmt} = \$400$ | The periodic payment is $400 per month |
| $\mathrm{r} =$ 5% or .05 | 5% annual compound interest rate |
| $\mathrm{t} =$ 30 years | the time was 30 years |
| $\mathrm{n} =$ 12 | There are 12 monthly payments, which is the same as the number of compounding per year |
| $\mathrm{FV} = \frac{400\left[\left(1+\frac{.05}{12}\right)^{12\cdot 30}-1\right]}{\left(\frac{.05}{12}\right)}$ | Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols. |
| $\mathrm{FV} = \$332{,}903.45$ | Carefully, push all the buttons on the calculator. |

Look below for the steps that will work with many scientific calculators.

The next question is how much of the money came from interest. To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

| Steps | Reasons |
|---|---|
| Deposits = 400·12·30<br>Deposits = $144,000 | To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 30 years. |
| Interest = future value – deposits<br>Interest = 332,903.45 – 144,000 | The interest is the amount of money at the end minus the amount of money deposited. |

The interest is $188,903.45

Many scientific calculators require us to push the following buttons:

**Formula**

$$\text{FV} = \frac{400\left[\left(1+\frac{.05}{12}\right)^{12\cdot 30}-1\right]}{\left(\frac{.05}{12}\right)}$$

**Buttons**

| | |
|---|---|
| 400 | |
| ( | open parentheses twice like the formula |
| ( | |
| 1 | |
| Plus | |
| .05 | |
| Divide | |
| 12 | |
| ) | close parentheses |
| ^ or $x^y$ | use the exponent button |
| 360 or $(12 \cdot 30)$ | type either way |
| Minus | |
| 1 | |
| ) | |
| Divide | |
| ( | |
| .05 | |
| Divide | |
| 12 | |
| ) | |
| = | |

To get out of the exponent area some calculators require pushing a right arrow.

2. A young person is deciding whether to start investing now or wait. After all, he is young, has many expenses like cars and a house, and wants to enjoy his money while being young.

a. If interest is 6% compounded monthly, how much money will the investor have if he invests $500 per month for 40 years? How much of that money comes from interest?

b. If the interest is kept at 6% compounded monthly, how much money will the investor have if he puts aside $1000 per month for 20 years? How much of that money will be interest?

a. Consider the first scenario.

| Steps | Reasons |
|---|---|
| $\mathrm{FV} = \dfrac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$ | Since the investor is putting aside a set amount each month for forty years at a constant interest rate, we have an annuity. |
| $\mathrm{FV} =$ ??? | We are looking for the accumulated value or future value of the annuity. |
| $\mathrm{Pmt} = \$500$ | The periodic payment is $500 per month |
| $\mathrm{r} = 6\%$ or .06 | Assuming 6% annual compound interest rate |
| $\mathrm{t} = 40$ years | the time was 40 years |
| $\mathrm{n} = 12$ | There are 12 monthly payments, which is the same as the number of compounding per year |
| $\mathrm{FV} = \dfrac{500\left[\left(1+\frac{.06}{12}\right)^{12\cdot 40}-1\right]}{\left(\frac{.06}{12}\right)}$ | Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols. |
| $\mathrm{FV} = \$995{,}745.37$ | Carefully, push all the buttons on the calculator as outlined above. |

Now find the interest.

| Steps | Reasons |
|---|---|
| Deposits = 500·12·40<br>Deposits = $240,000 | To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 40 years. |
| Interest = future value – deposits<br>Interest = 995,745.37 – 240,000 | The interest is the amount of money at the end minus the amount of money deposited. |

The interest is $755,745.37

b. Consider the second scenario.

| Steps | Reasons |
|---|---|
| $\mathrm{FV} = \dfrac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$ | Since he is putting aside a set amount each month for thirty years at a constant interest rate, we have an annuity. |
| $\mathrm{FV} =$ ??? | We are looking for the accumulated value or future value of the annuity. |
| $\mathrm{Pmt} = \$1000$ | The periodic payment is \$1000 per month |
| $\mathrm{r} = 6\%$ or .06 | $6\%$ annual compound interest rate |
| $\mathrm{t} =$ 20 years | the time is 20 years |
| $\mathrm{n}=$ 12 | There are 12 monthly payments, which is the same as the number of compounding per year |
| $\mathrm{FV} = \dfrac{1000\left[\left(1+\frac{.06}{12}\right)^{12\cdot 20}-1\right]}{\left(\frac{.06}{12}\right)}$ | Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols. |
| $\mathrm{FV} = \$462{,}040.90$ | Carefully, push all the buttons on the calculator as in example 1. |

To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

| Steps | Reasons |
|---|---|
| Deposits = 1000·12·20<br>Deposits = \$240,000 | To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 20 years. |
| Interest = future value – deposits<br>Interest = 462,040.90 – 240,000 | The interest is the amount of money at the end minus the amount of money deposited. |

The interest is \$222,040.90

In both the 20 year and the 40 year examples, the investor puts aside the same total amount of money (\$240,000). By starting sooner as in scenario a, the investor gains an extra \$533,704.47 all in interest.

## Periodic Deposits of an Annuity

We may want to how much money we need to set put aside regularly to reach a future financial goal. If we put the same restrictions that we had before of periodic payments with a constant interest rate, then we get a formula for the periodic payment based on a known goal. Most of us will be interested in our retirement and perhaps our children's education. If we know the future amount that we want, then we can calculate the amount that needs to be put aside regularly.

Formula for periodic payment of an annuity:

$$\mathrm{Pmt} = \frac{\mathrm{FV}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}} - 1\right]}$$

$\mathrm{Pmt}$ = periodic payment
$\mathrm{FV}$ = future value for an annuity
$\mathrm{r}$ = annual compound interest rate
$\mathrm{t}$ = time measured in years
$\mathrm{n}$= number of payments, which is the same as the number of compounding per year

Examples:

3. When studying compound interest, we saw that a twenty-five year old would have to put aside a bit over two hundred thousand dollars all at once for 40 years at 3.25% monthly interest to reach a retirement goal of $750,000 to retire comfortably. While that is an amazing amount of interest unfortunately many of us do not have that kind of money on hand. How much does the twenty-five year old need to put aside each month to have $750,000 in 40 years at 3.25% compounded monthly? How much of that amount is interest?

| Steps | Reasons |
|---|---|
| $\mathrm{Pmt} = \frac{\mathrm{FV}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}} - 1\right]}$ | Since we are putting aside a set amount each month for forty years at a constant interest rate, we have an annuity. |
| $\mathrm{Pmt} = ???$ | We are looking for the periodic (monthly) payment |
| $\mathrm{FV}$ = 750,000 | The future value of the annuity is the goal of $750,00 |
| $\mathrm{r}$ = 3.25% or .0325 | 3.25% annual compound interest rate |
| $\mathrm{t}$ = 40 years | The time was 40 years |
| $\mathrm{n}$= 12 | There are 12 monthly payments, which is the same as the number of compounding per year |
| $\mathrm{Pmt} = \frac{750{,}000\left(\frac{.0325}{12}\right)}{\left[\left(1+\frac{.0325}{12}\right)^{12\cdot 40} - 1\right]}$ | Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols. |
| $\mathrm{Pmt} = \$762.81$ | Carefully, push all the buttons on the calculator. |

The next question is how much of the money came from interest. To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

| Steps | Reasons |
|---|---|
| Deposits = 762.81·12·40<br>Deposits = $366,148.80 | To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 40 years. |
| Interest = future value – deposits<br>Interest = 750,000 – 366,148.80 | The interest is the amount of money at the end minus the amount of money deposited. |

The interest is $383,851.20

4. A company wants to make an extra retirement fund for its executives so that the CEO and other top officers have an extra $10,000,000 to share. How much money does the company need to put aside quarterly for 15 years at 5.25% interest compounded quarterly to meet this goal? How much of that goal is deposits and how much is interest?

| Steps | Reasons |
|---|---|
| $\text{Pmt} = \dfrac{\text{FV}\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}$ | Since we are putting aside a set amount each month for 15 years at a constant interest rate, we have an annuity. |
| Pmt = ??? | We are looking for the periodic (quarterly) payment |
| FV = 10,000,000 | The future value of the annuity is the goal of $10,000,00 |
| r = 5.25% or .0525 | 5.25% annual compound interest rate |
| t = 15 years | The time is 15 years |
| n= 4 | There are 4 quarterly payments, which is the same as the number of compounding per year |
| $\text{Pmt} = \dfrac{10{,}000{,}000\left(\frac{.0525}{4}\right)}{\left[\left(1+\frac{.0525}{4}\right)^{4\cdot 15} - 1\right]}$ | Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols. |
| Pmt = $110,604.24 | Carefully, push all the buttons on the calculator. |

The next question is how much of the money came from interest. To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

| Steps | Reasons |
|---|---|
| Deposits = 110,604.24·4·15<br>Deposits = $6,636,254.40 | To find the amount of money that is put into the annuity, we take the monthly payment multiply by 4 months per year and then multiply by 15 years. |
| Interest = future value – deposits<br>Interest = 10,000,000 – 6,636,254,4 | The interest is the amount of money at the end minus the amount of money deposited. |

The interest is $3,363,745.60

There are several ways to increase the future value of an annuity:
Higher interest
Larger deposits
Longer time investing
We cannot control the interest rates that we receive, but we can start saving for our retirement while we are young rather than waiting.

Exercises

For the following, round to the nearest dollar.

1. If $250 is put aside every month at 3.5% interest compounded monthly for 30 years, what is the accumulated value? How much of that accumulated value is interest?

2. If $450 is put aside every month at 7.5% interest compounded monthly for 25 years, what is the accumulated value? How much of that accumulated value is interest?

3. If $600 is put aside every month at 6.8% interest compounded monthly for 40 years, what is the accumulated value? How much of that accumulated value is interest?

4. If $520 is put aside every month at 3.2% interest compounded monthly for 35 years, what is the accumulated value? How much of that accumulated value is interest?

5. A young person is deciding whether to start investing now or wait. After all, she is young, has many expenses like cars and a house, and wants to enjoy her money while being young.
    a. If interest is 4.5% compounded monthly, how much money will the investor have if she invests $450 per month for 40 years? How much of that money comes from interest?
    b. If the interest is kept at 4.5% compounded monthly, how much money will the investor have if she puts aside $900 per month for 20 years? How much of that money will be interest?
    c. In both cases the same amount of money is put aside. Does the extra interest make it worth starting to save for retirement early?

6. A young person is deciding whether to start investing now or wait. After all, he is young, has many expenses like cars and a house, and wants to enjoy his money while being young.
    a. If interest is 5.8% compounded monthly, how much money will the investor have if he invests $600 per month for 40 years? How much of that money comes from interest?
    b. If the interest is kept at 5.8% compounded monthly, how much money will the investor have if he puts aside $1200 per month for 20 years? How much of that money will be interest?

c. In both cases the same amount of money is put aside. Does the extra interest make it worth starting to save for retirement early?

7. How much does a thirty year old need to put aside each month to have $1,000,000 in 35 years at 6.25% interest compounded monthly? How much of that amount is interest?

8. How much does a twenty year old need to put aside each month to have $1,500,000 in 45 years at 7.25% interest compounded monthly? How much of that amount is interest?

9. How much does a fifty year old need to put aside each month to have $1,000,000 in 15 years at 4.5% interest compounded monthly? How much of that amount is interest?

10. How much does an eighteen year old need to put aside each month to have $2,000,000 in 47 years at 5.25% interest compounded monthly? How much of that amount is interest?

11. A couple wants to save for their child’s education. Since education costs are always rising, they decide to try to save $200,000. If the child does not want to go to college, the couple figures that they can get a lake house.
    a. How much money does the couple need to put aside each month for 18 years at 5.25% interest compound monthly to reach their goal?
    b. How much money does the couple need to put aside each month for 9 years at 5.25% interest compound monthly to reach their goal?
    c. Is it important for the couple to start early to reach their goal and pay for their child’s education?
    d. How much money does the couple need to put aside each month for 18 years at 9.25% interest compound monthly to reach their goal? At 2.25% compounded monthly?

12. A couple wants to save for their child’s education. Since education costs are always rising, they decide to try to save $150,000. If the child does not want to go to college, the couple figures that they can use the money to help with their retirement.
    a. How much money does the couple need to put aside each month for 18 years at 4.25% interest compound monthly to reach their goal?
    b. How much money does the couple need to put aside each month for 9 years at 4.25% interest compound monthly to reach their goal?

c. Is it important for the couple to start early to reach their goal and pay for their child's education?
d. How much money does the couple need to put aside each month for 18 years at 8.25% interest compound monthly to reach their goal? At 3.25% compounded monthly?

13. A young bachelor feels that he spends the majority of his paycheck on tobacco, alcohol, and hitting the town with his friends. He figures that if he quits smoking, only goes out on the weekend, and limits dining out to special occasions, he will be able to save $350 per month. If he puts that money into an annuity that earns 4.75% interest compounded monthly, how much will he have saved after 35 years? How much of that money is interest?

14. A chronic gambler seeks financial help and discovers that she is losing $200 per month by playing the slot machines. If she saves $200 per month at 6.25% interest compounded monthly, how much will she have saved after 25 years? How much of that money comes from interest?

15. A company wants to make an extra retirement fund for its employees so that the employees who work for the company more than twenty years will get a large bonus upon retiring. If the company wants to save $20,000,000 in 10 years, how much money does the company need to put aside quarterly at 4.25% interest compounded quarterly to meet this goal? How much of the goal comes from the deposits and how much is interest?

16. A company wants to save $200,000,000 for future acquisitions. How much money must the company put aside quarterly for 15 years at 6.5% interest compounded quarterly? How much of the goal comes from the deposits and how much is interest?

For most of us our largest purchase will be a house. We may end up buying more than one house, but it is unlikely that we will buy more than two or three houses. The bank's loan officer will be the expert. He will be trained and may have processed hundred's of loans by the time you talk to him about your mortgage. Making sure that you are informed of the options and their consequences will be the first step in making sure that you get the best possible deal with the bank when you buy your house.

A mortgage is an amount of money borrowed over a certain amount of time, which is paid back with periodic payments. There are two types of mortgages: variable and fixed interest mortgages. Variable interest may offer a lower rate in the beginning, but that rate can change over the time of the mortgage depending on current economic conditions. If the mortgage rate goes up too much, the borrower may find himself in a situation where he cannot afford the payments. A fixed interest mortgage has a set interest rate for the life of the loan. The interest rate may be slightly higher in the beginning, but the borrower does not take the chance the interest rates increase along with the periodic payments.

Formula for periodic mortgage payments with fixed interest:

$$\mathrm{Pmt} = \frac{\mathrm{B}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[1 - \left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{-\mathrm{nt}}\right]}$$

$\mathrm{Pmt}$ = periodic payment
$\mathrm{B}$ = the amount of money borrowed
$\mathrm{r}$ = annual compound interest rate
$\mathrm{t}$ = time measured in years
$\mathrm{n}$= number of payments, which is the same as the number of compounding per year

The most important examples for us will involve the purchase of a house with monthly payments within our possibilities. We should want to know under what conditions we can save the most money. Some other important definitions for mortgages are down payment and points.

The down payment is a percent of the selling price, which is paid at the time the house is bought. Subtracting the down payment from the selling price will give us the amount borrowed, which is $\mathrm{B}$ in our above formula.

Points are a fee paid to the bank. Each point is equal to 1% of the amount borrowed. So, 2.5 points is 2.5% of the amount being borrowed. Points may be required by the bank. Sometimes points can be paid to get more favorable conditions such as a lower interest rate.

Examples:

1. A couple wanting to buy a house in the Midwest has finally found a house that they think is right for them. The house costs $300,000 and can be financed with a fixed rate mortgage of 30 years at 4.8% compounded monthly. They decide to make a 5% down payment, and 1.5 points must be paid to the bank at closing.

a. What is the down payment?

| Steps | Reasons |
|---|---|
| 5% of 300,000<br>.05(300,000) | Down payment is a percent of the selling price that is paid to the seller on the spot. |

The down payment is $15,000.

b. What is the amount borrowed?

| Steps | Reasons |
|---|---|
| 300,000 – 15,000<br>$285,000 is borrowed. | The couple needs to borrow the selling price minus the down payment, which is already paid to the seller. |

c. What is the price of the 1.5 points at closing?

| Steps | Reasons |
|---|---|
| 1.5% of 285,000<br>.015(285,000) | 1.5 points represents 1.5% of the amount borrowed. These points should be thought of as a bank fee. |

$4275 paid for points.

d. What are the monthly payments?

| Steps | Reasons |
|---|---|
| $\mathrm{Pmt} = \dfrac{B\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$ | We are paying back a loan with periodic (monthly) payments. |
| $\mathrm{Pmt}$ = ???<br>$B$ = $285,000<br>$r$ = 4.8% or .048<br>$t$ = 30 years<br>$n$= 12 | $\mathrm{Pmt}$ = we are looking for the monthly mortgage payment<br>The amount borrowed is $285,000. 4.8% annual compound interest rate. The time is 30 years.<br>There are 12 monthly payments, which is the same as the number of compounding per year. |

$$\text{Pmt} = \frac{285{,}000\left(\frac{.048}{12}\right)}{\left[1-\left(1+\frac{.048}{12}\right)^{-12\cdot 30}\right]}$$

Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.

$\text{Pmt} = \$1495.30$ Carefully, push all the buttons on the calculator.

The monthly mortgage payment is $1495.30.

e. How much interest is paid?

To get the interest, find out how much of the money was paid altogether and subtract the amount borrowed.

| Steps | Reasons |
|---|---|
| Total amount paid = 1495.30·12·30<br>Deposits = $538,308 | To find the total amount of money that is paid to the bank, we take the monthly payment multiply by 12 months per year and then multiply by 30 years. |
| Interest = total amount paid – amount borrowed<br>Interest = 538,308 – 285,000<br><br>The interest is $253,308. | The interest is the total amount of paid minus the amount borrowed. |

A quarter of a million dollars just in interest is a lot to pay. Think of saving a quarter of a million dollars and then handing it over to the bank. What can the couple do? Let's take a look at the couple buying the same house, but under some circumstances that will save them money.

2. The couple wants to buy the same house in the Midwest, which they think is just right for them. The house costs $300,000, but this time they will finance it with a fixed rate mortgage of 10 years at 4.3% compounded monthly. They decide to make a 25% down payment, and 2 points must be paid to the bank at closing. How much must be paid for the points at closing? What is the monthly mortgage payment? What is the total interest?

To find the amount paid for points and the monthly mortgage payment we need the amount borrowed. So, first we calculate the down payment and subtract it from the price of the house.

| Steps | Reasons |
|---|---|
| 25% of 300,000<br>.25(300,000) | Down payment is a percent of the selling price that is paid to the seller on the spot. |

The down payment is $75,000.

| | |
|---|---|
| 300,000 – 75,000 | The couple needs to borrow the selling price minus the down payment, which is already paid to the seller. |

$225,000 is borrowed.

| | |
|---|---|
| 2% of 225,000<br>.02(225,000) | 2 points represents 2% of the amount borrowed. These points should be thought of as a bank fee. |

$4500 paid for points.

| | |
|---|---|
| $\text{Pmt} = \dfrac{B\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$ | We are paying back a loan with periodic (monthly) payments. |
| $\text{Pmt}$ = ???<br>$B$ = $225,000<br>$r$ = 4.3% or .043<br>$t$ = 10 years<br>$n$= 12 | $\text{Pmt}$ = we are looking for the monthly mortgage payment<br>The amount borrowed is $225,000.<br>4.3% annual compound interest rate<br>The time is 10 years.<br>There are 12 monthly payments, which is the same as the number of compounding per year. |
| $\text{Pmt} = \dfrac{225{,}000\left(\frac{.043}{12}\right)}{\left[1-\left(1+\frac{.043}{12}\right)^{-12\cdot 10}\right]}$ | Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols. |
| $\text{Pmt}$ = $2310.23 | Carefully, push all the buttons on the calculator. |

The monthly mortgage payment is $2310.23

| | |
|---|---|
| Total amount paid =<br>2310.23·12·10<br>Deposits = $277,227.60 | To find the total amount of money that is paid to the bank, we take the monthly payment multiply by 12 months per year and then multiply by 10 years. |
| Interest = total amount paid–amount borrowed<br>Interest = 277,227.60 – 225,000 | The interest is the total amount of money paid minus the amount borrowed. |

The interest is $52,227.60.

In both examples the amount of money paid for points is about the same, but with the 10 year mortgage there is a huge savings in interest. With the 30 year mortgage in the first example the interest is $253,308. In the second example with a 10 year mortgage, the interest is $52,227.60. That means that the couple saves over $200,000 with the 10 year mortgage. By paying a larger down payment with a shorter borrowing time, you too can save more money than you may have thought. Remember to ask the loan officer what will be the monthly payment for a 10 year, 20 year, and 30 year loans. Calculate what the total interest will be for the life of the loan for various scenarios. Try to save up so that you have a larger down payment even if the bank does not require it.

Many other loans work like mortgages. Loans to buy a car, a boat, a kitchen appliance, or just about anything else may work like a mortgage. Although you will not pay points, you may pay a down payment with monthly payments.

Example:
3. A fisherman wants to buy a top of the line boat that costs $30,000. To help him pay for it, the boat company offers to let him pay a 10% down payment with monthly payments at 8% interest compounded monthly for 10 years. What are the monthly payments? How much interest is paid to finance the boat?

To find the monthly boat payment we need the amount borrowed. So, first we calculate the down payment and subtract it from the price of the boat.

| Steps | Reasons |
|---|---|
| 10% of 30,000<br>.10(30,000) | Down payment is a percent of the selling price that is paid to the seller on the spot. |

The down payment is $3,000.

| | |
|---|---|
| 30,000 – 3,000<br><br>$27,000 is borrowed. | The fisherman will borrow the selling price of the boat minus the down payment, which is already paid to the seller. |
| $\text{Pmt} = \dfrac{B\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$ | We need to find the monthly payments. So, we use the formula for periodic payments with fixed interest. |
| $\text{Pmt}$ = ??? | $\text{Pmt}$ = we are looking for the monthly boat payment |
| $B$ = $27,000 | The amount borrowed is $27,000. |
| $r$ = 8% or .08 | 8% annual compound interest rate |
| $t$ = 10 years | The time is 10 years. |
| $n$= 12 | There are 12 monthly payments per year, which is the same as the number of compounding per year. |

$$\text{Pmt} = \frac{27{,}000\left(\frac{.08}{12}\right)}{\left[1-\left(1+\frac{.08}{12}\right)^{-12\cdot 10}\right]}$$

Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.

$\text{Pmt} =$ $327.58

Carefully, push all the buttons on the calculator.

<u>The monthly boat payment is $327.58</u>

Total amount paid =
327.58·12·10
Deposits = $39,309.60

To find the total amount of money that is paid, we take the monthly payment multiply by 12 months per year and then multiply by 10 years.

Interest = total amount paid–amount borrowed
Interest = 39,309.60 – 27,000

The interest is the total amount of money paid minus the amount borrowed.

<u>The interest is $12,309.60.</u>

Here the fisherman is paying an extra $9,309.60 in interest to buy the boat. If we finance our purchases at high interest rates, we will always end up paying more for the items that what we buy.

Exercises

For the following questions, round to the nearest dollar.

1. A couple wants to buy a house that costs $275,000. The house can be financed with a fixed rate mortgage of 30 years at 5.25% compounded monthly. They decide to make a 5% down payment, and 1.5 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 1.5 points at closing?
   d. What are the monthly payments?
   e. How much interest is paid?

2. A couple wants to buy a house that costs $320,000. The house can be financed with a fixed rate mortgage of 15 years at 4.8% compounded monthly. They decide to make a 25% down payment, and 2.5 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 2.5 points at closing?
   d. What are the monthly payments?
   e. How much interest is paid?

3. A doctor wants to buy a house that costs $580,000. The house can be financed with a fixed rate mortgage of 15 years at 4.25% compounded monthly.
   She decides to make a 30% down payment, and 2 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 2 points at closing?
   d. What are the monthly payments?
   e. How much interest is paid?

4. A bachelor wants to buy an apartment that costs $110,000. The apartment can be financed with a fixed rate mortgage of 30 years at 5.75% compounded monthly. He decides to make a 5% down payment, and 1.5 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 2.5 points at closing?

d. What are the monthly payments?
e. How much interest is paid?

5. A couple is trying to decide between two mortgage options to finance the cost of a $325,000 house. Find the monthly payments and the total interest paid for both options.
   a. The couple pays a 5% down payment and finances the house with a 30 year mortgage at 5.75% interest. No points are paid at closing.
   b. The couple pays a 25% down payment and finances the house with a 10 year mortgage at 5.25% interest. No points are paid at closing.
   c. Which option do you think is better for the couple?

6. A lawyer is trying to decide between two mortgage options to finance the cost of a $625,000 house. Find the monthly payments and the total interest paid for both options.
   a. The lawyer pays a 5% down payment and finances the house with a 30 year mortgage at 6.75% interest. No points are paid at closing.
   b. The lawyer pays a 30% down payment and finances the house with a 15 year mortgage at 6.25% interest. No points are paid at closing.
   c. Which option do you think is better for the lawyer?

7. A bachelor is trying to decide between two mortgage options to finance the cost of a $225,000 house. Find the monthly payments and the total interest paid for both options.
   a. The bachelor pays no down payment and finances the house with a 40 year mortgage at 6.25% interest. No points are paid at closing.
   b. The bachelor pays a 20% down payment and finances the house with a 15 year mortgage at 5.50% interest. No points are paid at closing.
   c. Which option do you think is better for the bachelor?

8. A couple is trying to decide between two mortgage options to finance the cost of a $350,000 house. Find the monthly payments and the total interest paid for both options.
   a. The couple pays a 5% down payment and finances the house with a 40 year mortgage at 5.9% interest. No points are paid at closing.
   b. The couple pays a 25% down payment and finances the house with a 10 year mortgage at 5.2% interest. No points are paid at closing.
   c. Which option do you think is better for the couple?

9. A fisherman wants to buy a top of the line boat that costs $40,000. To help him pay for it, the boat company offers to let him pay a 15% down payment with monthly payments at 9% interest compounded monthly for 10 years. What are the monthly payments? How much interest is paid to finance the boat?

10. To buy a car rather than paying cash the buyer considers financing it. For a $45,000 convertible the car company offers the individual a no down payment option with 5 year financing at a 7% interest rate. What are the monthly payments and what is the total interest paid to finance the purchase of the car?

11. To buy a car, rather than paying cash the buyer considers financing it. For a $37,500 convertible the car company offers the individual a 20% down payment option with 3 year financing at a 3% interest rate. What are the monthly payments and what is the total interest paid to finance the purchase of the car?

12. A student lost his funding for college and decided to finance his last two years largely by taking on credit card debt. When the student finally got his degree, he found out that he owed $65,000 to various credit cards. To pay off his debts the student destroyed his credit cards and consolidated his credit card debt. He took out a 10 year loan at 11.5% compounded monthly. How much are his monthly payments and what is the total interest that needs to be paid?

13. To pay off hospital bills a worker with poor credit has to take out a 10 year loan. If the worker borrows $60,000 at 9.5% compounded monthly, what will be the monthly payments and the total interest paid for the loan?

14. A student that goes to college finishes with $115,000 debt. The student's loan is for 15 years at 8.5% interest compounded monthly. What are the monthly payments and what is the total interest paid for the loan?

# 7

Understanding how luck works goes a long way towards understanding the random events in the universe. Instead of being completely random it turns out that random events follow rules of probability, which depend on the conditions under which the events occur. Whether you are trying to win big in Las Vegas or simply count how many ways certain situations can occur understanding the basics is the beginning.

Course Outcomes:

- Calculate probabilities, and use and apply the normal distribution (Note that the normal distribution will be covered in Chapter 8.)

7.1 Counting Rules

The fundamental counting rule, permutations, and combinations are covered. Successful students will be able to determine the appropriate counting rule and calculate the result by hand or using a calculator.

7.2 Probabilities

The types of probabilities (subjective, empirical, and theoretical) are introduced. Theoretical probabilities and associated definitions are studied in depth. Contingency tables are shown.

7.3 Complement Rule and Addition Rule

Students need to focus on important aspects of the question to determine the appropriate probability rule. Mutually exclusive events are defined and used to determine the appropriate addition rule formula. Contingency tables are further developed as pertains to the addition rule.

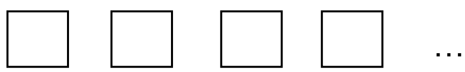
7.4 Multiplication Rule and Conditional Probabilities

Two more probability rules are studied. Independent and dependent events are defined and used to determine how to apply the multiplication rule. Conditional probabilities are explained by restricting the sample space according to the given condition. Contingency tables are reviewed and further developed as pertains to conditional probabilities.

Have you ever wondered how many different lottery tickets there are? Or, how many ways there are to pick people from a group to do a job? We often study these types of counting situations before we study probabilities. Here we will look at three counting rules and the good news is that we will always be asking, “How many …? or how many ways …?”

Fundamental Counting Rule

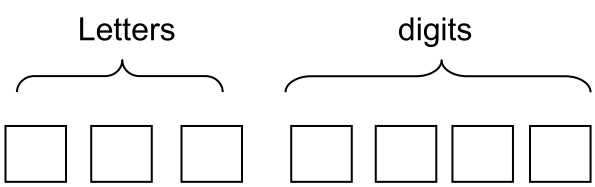
The number of ways that a sequence (or list) can occur is equal to the product of the number of ways that each element in the sequence can occur.



$$k_1 \cdot k_2 \cdot k_3 \cdot k_4 \cdot \ldots$$

Examples:

1. How many different license plates are there with three letters followed by four digits?




$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

There are 26 letters and 10 digits (0,1,2,3,4,5,6,7,8,9)

So, there are $26^3 \cdot 10^4 = 175{,}760{,}000$ different license plates.

Looking at the last four digits helps us to see where the counting rule comes from. Four digits could be

0000 , 0001 , 0002 , 0003, … , 9997 , 9998 , 9999

That would make 9999 + 1 (because of 0000) or 10,000 possibilities, which is the same as $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10{,}000$

2. How many different ways are there to have three children if we are considering whether the children are boys or girls?

$$2 \cdot 2 \cdot 2 = 8$$

There are 8 possible boy/girl possibilities for 3 children

3. A new car can be bought with three types of options – color, motor, level of luxury. If there are five color options, three motor options, and two levels of luxury (basic or deluxe), how many different types of cars are there with these options?

color motor luxury

$$5 \cdot 3 \cdot 2 = 30$$

There are 30 ways to buy the car.

Factorials

Before talking about the next two counting rules, it will be useful to understand factorials. For a whole number $n$, $n$ factorial written $n!$ means:

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$

So,

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$1! = 1$

$0! = 1$ This one is a bit strange.

Examples:

4. $\frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 = 210$

If we are doing factorials, cancelling common factors may be helpful.

5. $\frac{800!}{799!} = \frac{800 \cdot \cancel{799} \cdot \cancel{798} \cdot \cancel{797} \cdot \cancel{796} \cdot \cancel{795} \cdot \ldots}{\cancel{799} \cdot \cancel{798} \cdot \cancel{797} \cdot \cancel{796} \cdot \cancel{795} \cdot \ldots} = 800$

Here we really need to cancel the factors because $800!$ and $799!$ are too large to fit on our calculators. Basic scientific calculators do include a factorial button, !, that will calculate the factorials for us.

Permutations:

A permutation is a subset of a larger set where order matters. When we say order matters what we really want to say is that changing the members of the subset gives us a different situation. For instance horses coming in first, second, or third in a race of 10 horses is a permutation because changing which horse comes in first, second, and third gives us a different situation. Another example of a permutation is selecting president and vice president from a board of 15 members. Switching who is the president and who is the vice president gives another way to select the president and vice president.

To calculate the number of permutations we can use a formula or a scientific calculator. Using a scientific calculator to do the complete calculation is easier.

Formula for Permutation:

$${}_{n}P_{k} = \frac{n!}{(n-k)!}$$

$n$ = the size of the larger set from which smaller set is chosen
$k$ = the size of the smaller subset

Examples:

6. How many ways can swimmers come in first, second, and third in a race of 12 swimmers?

| Steps | Reasons |
| --- | --- |
| ${}_{12}P_{3}$ | Permutation: there is a subset of three racers that are taken from the twelve racers and the order matters. |
| ${}_{12}P_{3} = \frac{12!}{(12-3)!}$ | Formula: ${}_{n}P_{k} = \frac{n!}{(n-k)!}$ |
| $= \frac{12!}{9!}$ <br> $= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \ldots}{9\cdot 8\cdot 7\cdot \ldots}$ <br> $= 12 \cdot 11 \cdot 10$ | Calculate using the formula or skip this step by using a scientific calculator. |

There are 1320 ways swimmers can come in first, second, and third in a race.

7. How many ways to choose a President, Vice President, Secretary, and Treasurer from a class of twenty students?

| Steps | Reasons |
|---|---|
| ${}_{20}P_4$ | Permutation: there is a subset of four officers that are taken from the twenty students and the order matters because they have different jobs. |
| ${}_{20}P_4 = \frac{20!}{(20-4)!}$ | Formula: ${}_{n}P_k = \frac{n!}{(n-k)!}$ |
| $= \frac{20!}{16!}$ <br> $= \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot \ldots}{16\cdot 15\cdot 14\cdot \ldots}$ <br> $= 20\cdot 19\cdot 18\cdot 17$ | Calculate using the formula or skip this step by using a scientific calculator. |

There are 116,280 ways to choose a President, Vice President, Secretary, and Treasurer from a class of twenty students.

8. A lottery has one $50 prize and another $10 prize. If fifty people each buy one ticket, how many ways are there for the prizes to be awarded?

| Steps | Reasons |
|---|---|
| ${}_{50}P_2$ | Permutation: there is a subset of two prize winners that are taken from the fifty people that buy tickets. Order matters because the prize values are different. |
| ${}_{50}P_2 = \frac{50!}{(50-2)!}$ | Formula: ${}_{n}P_k = \frac{n!}{(n-k)!}$ |
| $= \frac{50!}{48!}$ <br> $= \frac{50\cdot 49\cdot 48\cdot 47\cdot 46\cdot \ldots}{48\cdot 47\cdot 46\cdot \ldots}$ <br> $= 50\cdot 49$ | Calculate using the formula or skip this step by using a scientific calculator. |

There are 2450 ways to award the two prizes.

## Combinations:

A combination is a subset of a larger set where order does not matter. When we say "order does not matter" what we really want to say is that changing the members of the subset gives us the same situation. For instance selecting a committee of five members from a group of forty senators is a combination because we are taking a subset of five from the larger group of forty and the order is not important because the committee members cannot be distinguished in any way.

Formula for Combination:

$${}_{n}C_{k} = \frac{n!}{(n-k)!\,k!}$$

n = the size of the larger set from which smaller set is chosen
k = the size of the smaller subset

To calculate the number of combinations we can use a formula or a scientific calculator. Using a scientific calculator to do the complete calculation is easier.

Examples:

9. From the explanation of combinations above, how many are there to select a committee of five members from a group of forty senators?

| Steps | Reasons |
|---|---|
| ${}_{40}C_{5}$ | Combination: there is a subset of five members on the committee and order does not matter because they have the same job. |
| ${}_{40}C_{5} = \frac{40!}{(40-5)!\,5!}$ | Formula: ${}_{n}C_{k} = \frac{n!}{(n-k)!k!}$ |
| $= \frac{40!}{35!5!}$ $= \frac{40\cdot39\cdot38\cdot37\cdot36\cdot35\cdot34\cdot33\cdot\ldots}{(35\cdot34\cdot33\cdot\ldots)(5\cdot4\cdot3\cdot2\cdot1)}$ $= \frac{40\cdot39\cdot38\cdot37\cdot36}{5\cdot4\cdot3\cdot2\cdot1}$ $= 658{,}008$ | Calculate using the formula or skip this step by using a scientific calculator. |

There are 658,008 ways to select a committee of 5 out of 40 senators.

10. How many different five-card poker hands are there in a deck of fifty-two cards?

| Steps | Reasons |
|---|---|
| ${}_{52}C_{5}$ | Combination: there is a subset of five cards and the order that the cards are dealt does not matter. We only need to know the final cards we hold and we may very well move them around in our hand. |
| $2{,}598{,}960$ | Use a calculator with a combination button. |

There are 2,598,960 different poker hands.

11. How many different possible samples of 10 students from a group of 200 students?

| Steps | Reasons |
|---|---|
| ${}_{200}C_{10}$ | Combination: there is a subset of 10 students and the order that we select the students does not matter. The students do not have any jobs that will distinguish them from each other. If we list the names of the sample of ten students, the order that we write their names does not change anything. |
| $2.2451 \; x \; 10^{16}$ | Use a calculator with a combination button. |

There are 22,451,004,300,000,000 different possible samples!

12. How many ways to choose 3 carpenters and 4 painters out of 7 carpenters and 12 painters?

Here we have something a little different. We have the combinations of carpenters and the combinations of painters. We will use the Fundamental Counting Rule to multiply the two combinations.

| number of ways to choose the carpenters | number of ways to choose the painters |
|---|---|

$${}_{7}C_{3} \quad \cdot \quad {}_{12}C_{4} \quad = \quad 35 \cdot 495 = 17{,}325$$

There are 17,325 ways.

Exercises

Evaluate the following factorial problems:

1. $4!$

2. $7!$

3. $0!$

4. $5!$

5. $\frac{14!}{12!}$

6. $\frac{20!}{16!}$

7. $\frac{800!}{798!}$

8. $\frac{2000!}{1999!}$

9. $\frac{8!}{6!\cdot 2!}$

10. $\frac{10!}{3!\cdot 7!}$

11. $\frac{20!}{7!\cdot 3!}$

12. $\frac{15!}{4!\cdot 11!}$

Calculate the following using the Fundamental Counting Rule:

13. How many license plates are there that have two letters followed by five digits? Write two possible license plates.

14. How many license plates are there that have three letters followed by three digits? Write two possible license plates.

15. A diner has several breakfast options. A customer has three drink options (coffee, tea, or juice), five breakfast options (pancakes, waffles, omelet, fruit plate, or cereal), and three seating options (booth, table, or counter). How many different ways are there for a customer to enjoy breakfast? Write two possible breakfast options.

16. A diner has several lunch options. A customer has four drink options (coffee, tea, soda, or water), six food options (sandwich, salad, soup, meat, chicken, or fish), and three seating options (booth, table, or counter). How many different ways are there for a customer to enjoy lunch? Write two possible lunch options.

Calculate the following permutations. Using a calculator with a permutation button is easier than using the formula.

17. How many ways can horses come in first and second in a race with ten horses?

18. How many ways can sprinters come in first, second, and third in a race with ten sprinters?

19. How many ways are there for a class of 20 students to elect a president, vice president, secretary, and treasurer?

20. How many ways are there for a club with 15 members to elect a president, vice president, and treasurer?

Calculate the following combinations. Using a calculator with a combination button is easier than using the formula.

21. How many ways are there for a group of 18 students to select a committee of 4 members?

22. How many ways are there for a group of 10 professors to select a committee of 5 members?

23. From a group of 25 students how many ways are there to select a sample of 6 students to gain detailed feedback about a class?

24. From a group of 20 employees how many ways are there to select a sample of 8 employees to gain detailed feedback about working conditions?

25. How many ways to choose 2 carpenters and 5 painters out of 10 carpenters and 8 painters to renovate a house?

26. How many ways to choose 2 plumbers and 3 electricians out of 4 plumbers and 5 electricians to redo the wiring and plumbing in an old warehouse?

Calculate the following using the appropriate counting rule. Using a calculator with permutation and combination buttons where appropriate is easier than using the formulas.

27. An animal shelter has a variety of different dogs. A dog could be male or female. It could be large, medium, or small. The dog could be a pure breed or mixed. It could be a puppy, middle aged, or an older dog. How many different types of dogs can be adopted from the animal shelter if all of the above options are available?

28. From a class of 15 students, the professor asks 4 students for help moving the desks. How many ways are there for the professor to ask the students for help?

29. How many different ways are there to have five children if we are considering whether the children are boys or girls?

30. How many different ways are there to have six children if we are considering whether the children are boys or girls?

31. From a class of 15 students, the professor asks 4 students for help. One student will move desks, another student will take out the trash, a third student will clean out the bathrooms, and the fourth student will turn out the lights. How many ways are there for the professor to ask the students for help?

32. Five students decide to go see a concert. When they get back from the concert on Tuesday, they realize that they missed their math test. They go to the instructor and ask for a retest explaining that they had a flat tire and the spare was flat as well. When they open their make-up test, they see that there is only one question which is 'Which tire was flat?' How many ways could the five students answer which tire was flat?

33. How many ways are there to choose a President, Vice President, Secretary, and Treasurer from a class with 30 students?

34. A music club offers four free downloads as a promotion. If there are thirty songs to choose from, how many ways are there to download the four free promotional songs?

35. A chess club has a raffle to raise money. First prize is a bicycle, second prize is a chess set, and third prize is a $20 gift certificate. If 200 tickets are sold, how many ways are there for the prizes to be awarded?

36. To win the grand prize in a lottery somebody needs to pick five different numbers 1 to 60 as well as get the correct "series." The numbers cannot be repeated and there are 20 possibilities for the series. How many different lottery possibilities are there?

37. A sewing club has a raffle to raise money. Three brand new sewing machines are to be awarded. If 150 tickets are sold, how many ways are there for the prizes to be awarded?

38. In a math department there are 9 women and 6 men. How many ways are there to select 4 women and 2 men to represent the department at a conference?

39. In an English department there are 8 women and 15 men. How many ways are there to select 3 women and 5 men to represent the department at a conference?

40. A business man has three suits, five shirts, four ties, and two pairs of shoes. How many different ways can the businessman dress for a conference?

41. A beach lover has eight tee-shirts, four bathing suits, and three pairs of sandals. How many ways can he dress to go to the beach?

42. In order to make a fruit smoothie there are nine different types of fruit available. How many ways are there to choose four different types of fruit to make the smoothie?

43. A child wants to order a triple scoop ice cream cone at an ice cream shop. If there are twenty different types of ice cream to choose from and the child cares about the order in which the ice cream is stacked, how many different ways are there for the child to order the triple scoop ice cream cone?

44. From a group of thirty candidates, how many ways are there to select four of them to become Master Chief?

45. There are one hundred people that are given an experimental medicine. How many ways are there to select a sample of fifteen of them for further testing?

46. There are fifty entrees for an event in the Olympics. How many ways can the gold, silver, and bronze medals be awarded if it is impossible that two entrees receive the same medal?

47. A school lottery has one $500 prize and another $100 prize. If eighty people buy one ticket, how many ways are there for the two prizes to be awarded?

48. There are 7 boys and 10 girls in a high school class. How many ways are there to choose 2 boys and 3 girls to represent the class?

49. There are 15 doctors and 9 lawyers at a professional meeting on malpractice. How many ways are there to select 2 doctors and 3 lawyers to present at the meeting?

Probabilities are the likelihood of the occurrence of events. An outcome is an individual result of a probability experiment. An event is a collection of one or more outcomes. For instance if a fair die is rolled, there will be equal chances of rolling a 1, 2, 3, 4, 5, or 6. The 1, 2, 3, 4, 5, or 6 are the outcomes. The list of all possible outcomes is called the sample space. If we want to find the probability of rolling an odd number, then rolling an odd number is the event. The probability of rolling an odd number is $\frac{3}{6}$ or $\frac{1}{2}$ .

There are three notions of probabilities, but we will only study the theoretical probabilities in depth.

1. Subjective probabilities refer to hunches or opinions about the likelihood of an event occurring. For example, a student may think there is an 80% chance that he will have fun in an economics class.

2. Empirical (or observational) probabilities say that the probability that an event will occur is the number of times that the event was observed divided by the total number of observations. For instance if I flip a coin 100 times and get heads 60 times, empirical probabilities say that the chance of getting heads when flipping the coin is $\frac{60}{100}$ or 60%.

3. Theoretical Probabilities depend on specific assumptions, which then will allow us to apply rules. Given equally likely outcomes, the probability of an event occurring is equal to the number of ways the event can occur divided by the total number of possible outcomes. For example if I flip a fair coin, the probability of getting heads is $\frac{1}{2}$ because there is one outcome of head out of two possible outcomes of head or tail.

We will only study theoretical probabilities.

Basic Probability Rule:

Assume that all possible outcomes are equally likely. Then the probability of an event E occurring is:

$$\mathrm{P(E)} = \frac{\text{number of ways that E can occur}}{\text{total number of possible outcomes}}$$

The symbols "$\mathrm{P(E)}$" mean “the probability of event E occurring.”

Examples:

1. For a-e, consider a jar with 10 tokens as follows:

    5 red tokens numbered 1, 2, 3, 4, and 5
    3 green tokens numbered 1, 2, and 3
    2 yellow tokens numbered 1 and 2

All the tokens have the same chance of being selected. **One token is randomly selected.**

a. What is the probability of selecting a red token?

There are 5 red tokens.

$$\text{P(red token)} = \frac{\text{number of ways that red can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(red token)} = \frac{5}{10} \text{ or } \frac{1}{2} \text{ or } .5 \text{ or } 50\%$$

b. What is the probability selecting a number 2 token?

There are 3 tokens with number 2, one for each color.

$$\text{P(number 2)} = \frac{\text{number of ways that number 2 can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(number 2)} = \frac{3}{10} \text{ or } .3 \text{ or } 30\%$$

c. What is the probability of a token greater than 3?

There are 2 tokens with number greater than 3. The red 4 and red 5 are greater than 3.

$$\text{P(number greater than 3)} = \frac{\text{number of ways that greater than 3 can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(number greater than 3)} = \frac{2}{10} \text{ or } \frac{1}{5} \text{ or } .2 \text{ or } 20\%$$

d. What is the probability of a yellow token?

There are 2 yellow tokens.

$$\text{P(yellow tokens)} = \frac{\text{number of ways that greater than 3 can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(yellow tokens)} = \frac{2}{10} \text{ or } \frac{1}{5} \text{ or } .2 \text{ or } 20\%$$

e. What is the probability of a token that has a number of 6 or more?

There are no tokens with a number of 6 or more.

$$\text{P(number 6 or more)} = \frac{\text{number of ways that 6 or more can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(number 6 or more)} = \frac{0}{10} \text{ or } 0 \text{ or } 0\%$$

Notice that when an event cannot occur, it has a probability of 0. The maximum probability is 1 or 100%. In the example above, all ten numbered tokens form the sample space, which is the list of all possible outcomes. Each individual token is an outcome or individual result. The questions are about events.

2. For a-c, consider all of the equally likely outcomes for flipping a fair coin four times. The sample space (list of all possible outcomes is the following):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| HHHH | HHHT | HHTH | HHTT | HTHH | HTHT | HTTH | HTTT |
| THHH | THHT | THTH | THTT | TTHH | TTHT | TTTH | TTTT |

HHTH means heads on the first, second, and fourth tosses and tails on the third toss.

a. Find the probability of getting exactly one head.

Exactly one head can occur by HTTT, THTT, TTHT, or TTTH.

$$\mathrm{P(exactly\ one\ head)} = \frac{\text{number of ways that exactly one head can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(exactly\ one\ head)} = \frac{4}{16} \ \text{ or } \ \frac{1}{4} \ \text{ or } .25 \ \text{ or } \ 25\%$$

b. Find the probability of getting at most 3 heads.

Remember that at most three means three or less. At most three heads can occur in any of the following ways:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | HHHT | HHTH | HHTT | HTHH | HTHT | HTTH | HTTT |
| THHH | THHT | THTH | THTT | TTHH | TTHT | TTTH | TTTT |

$$\mathrm{P(at\ most\ three\ heads)} = \frac{\text{number of ways that at most 3 heads can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(at\ most\ three\ heads)} = \frac{15}{16} \ \text{ or } \ .9375 \ \text{ or } \ 93.75\%$$

c. Find the probability of getting exactly two tails.
Exactly two tails can occur in the following six ways:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | HHTT | | HTHT | HTTH | |
| | THHT | THTH | | TTHH | | | |

$$\mathrm{P(exactly\ two\ tails)} = \frac{\text{number of ways that exactly two tails can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(exactly\ two\ tails)} = \frac{6}{16} \ \text{ or } \ .375 \ \text{ or } \ 37.5\%$$

Another type of problem we can do involves a chart. We will start by using the same Basic Probability Rule as above.

3. For a-c, learners were classified according to motivation and performance. Motivation could be low, average, or high. Performance could be poor or good. The results are in the chart:

| | | Motivation | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Performance | Poor | 14 | 8 | 3 |
| | Good | 2 | 6 | 7 |

a. One learner is randomly selected from this group of learners. Find the probability that the learner has good performance.

**Whenever we have one of these chart problems the first thing we do is extend the chart by including totals.**

| | | Motivation | | | |
|---|---|---|---|---|---|
| | | Low | Average | High | Totals |
| Performance | Poor | 14 | 8 | 3 | 25 |
| | Good | 2 | 6 | 7 | 15 |
| | Totals | 16 | 14 | 10 | 40 |

$$\text{P(learner has good performance)} = \frac{\text{number of ways that good performance can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(learner has good performance)} = \frac{15}{40} \text{ or } \frac{3}{8} \text{ or } .375 \text{ or } 37.5\%$$

b. One learner is randomly selected from this group of learners. Find the probability that the learner has high motivation.

$$\text{P(learner has high motivation)} = \frac{\text{number of ways that high motivation can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(learner has high motivation)} = \frac{10}{40} \text{ or } \frac{1}{4} \text{ or } .25 \text{ or } 25\%$$

c. One learner is randomly selected from this group of learners. Find the probability that the learner has average motivation and good performance.

$$\text{P(average motiv. and good perf.)} = \frac{\text{number of ways that average motiv. and good perf. can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(average motiv. and good perf.)} = \frac{6}{40} \text{ or } \frac{3}{20} \text{ or } .15 \text{ or } 15\%$$

Exercises

Calculate the following probabilities:

Consider a six sided die with six equally likely outcomes: 1,2,3,4,5,6

1. What is the probability of rolling an even number?
2. What is the probability of rolling an odd number?
3. What is the probability of rolling less than 5?
4. What is the probability of rolling more than 5?
5. What is the probability of rolling at least 5?
6. What is the probability of rolling at most 5?
7. What is the probability of rolling less than 1?
8. What is the probability of rolling more than 6?

Consider all of the equally likely outcomes for flipping a fair coin three times where H represents a head and T represents a tail. The order represents the order of the flips. So, THH means that the first flip is tails while the second and third flips are heads. The sample space (list of all possible outcomes) is the following:

HHH HHT HTH HTT THH THT TTH TTT

9. What is the probability of getting exactly two heads?
10. What is the probability of getting exactly three tails?
11. What is the probability of getting at least two heads?
12. What is the probability of getting more than two tails?
13. What is the probability of getting exactly zero heads?
14. What is the probability of getting exactly one tail?
15. What is the probability of getting at most two tails?

16. What is the probability of getting at least one tail?

17. What is the probability of getting more than four tails?

18. What is the probability of getting less than zero heads?

Consider all of the equally likely outcomes for a family that has four children where B represents a boy and G represents a girl. The order represents the position in the family. So, BGGG means that the first child is a boy and the second, third, and fourth children are girls. The sample space (list of all possible outcomes is the following):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGGG | GGGB | GGBG | GGBB | GBGG | GBGB | GBBG | GBBB |
| BGGG | BGGB | BGBG | BGBB | BBGG | BBGB | BBBG | BBBB |

19. What is the probability of the family having exactly two boys?

20. What is the probability of the family having at least one girl?

21. What is the probability of the family having at most two girls?

22. What is the probability of the family having exactly three girls?

23. What is the probability of the family having exactly one boy?

24. What is the probability of the family having at most one girl?

25. What is the probability of the family having at least three boys?

26. What is the probability of the family having less than four girls?

27. What is the probability of the family having at least two girls?

28. What is the probability of the family having more than five girls?

29. What is the probability of the family having less than zero boys?

Consider a jar with 10 tokens as follows:

4 orange tokens numbered 1, 2, 3, and 4
3 blue tokens numbered 1, 2, and 3
3 red tokens numbered 1, 2, and 3

30. What is the probability of selecting a blue token?

31. What is the probability of selecting a number 1 token?

32. What is the probability of a token greater than 3?

33. What is the probability of a red token?

34. What is the probability of selecting an even numbered token?

Consider a jar with 15 tokens as follows:
6 yellow tokens numbered 1, 2, 3, 4, 5, and 6
3 blue tokens numbered 1, 2, and 3
6 green tokens numbered 1, 2, 3, 4, 5, and 6

35. What is the probability of selecting an even numbered token?

36. What is the probability selecting a number 6 token?

37. What is the probability of a token greater than 2?

38. What is the probability of a yellow token?

39. What is the probability of a blue token?

For the following, consider all the possible outcomes for rolling a fair die twice. The first number represents the first roll and the second number represents the second roll. For example 1,3 means that the first roll is 1 and the second roll is 3.

| | | | | | |
|---|---|---|---|---|---|
| 1,1 | 1,2 | 1,3 | 1,4 | 1,5 | 1,6 |
| 2,1 | 2,2 | 2,3 | 2,4 | 2,5 | 2,6 |
| 3,1 | 3,2 | 3,3 | 3,4 | 3,5 | 3,6 |
| 4,1 | 4,2 | 4,3 | 4,4 | 4,5 | 4,6 |
| 5,1 | 5,2 | 5,3 | 5,4 | 5,5 | 5,6 |
| 6,1 | 6,2 | 6,3 | 6,4 | 6,5 | 6,6 |

40. What is the probability that the sum of the two rolls is 4?

41. What is the probability that the sum of the two rolls is even?

42. What is the probability that the sum of the two rolls is at least 9?

43. What is the probability that the sum of the two rolls is less than 8?

44. What is the probability that the sum of the two rolls is 9?

45. What is the probability that the sum of the two rolls is at most 5?

46. What is the probability that the sum of the two rolls is more than 5?

47. What is the probability that the sum of the two rolls is more than 12?

48. What is the probability that the sum of the two rolls is at most 12?

Students are categorized by their attitude towards algebra and their ability to do statistics in the chart below. The students were asked if they like or dislike algebra and statistics ability was measured by their performance in an introductory statistics class. The results are in the chart:

| | | Statistics Ability | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Algebra | Likes | 5 | 12 | 10 |
| Attitude | Dislikes | 7 | 20 | 6 |

One student is randomly selected:

49. What is the probability that a randomly selected student dislikes algebra?

50. What is the probability that a randomly selected student has high statistics ability?

51. What is the probability that a randomly selected student likes algebra?

52. What is the probability that a randomly selected student has average statistics ability?

53. What is the probability that a randomly selected student has high statistics ability and likes algebra?

54. What is the probability that a randomly selected student has low statistics ability and likes algebra?

55. What is the probability that a randomly selected student likes algebra and has average statistics ability?

Dog owners that own only one dog are categorized by the size of the dogs that they have and whether the owners are female, male, or a couple. The results are in the chart:

| | | Dog Size | | |
|---|---|---|---|---|
| | | Small | Medium | Large |
| | Couple | 15 | 14 | 9 |
| Owners | Female | 11 | 12 | 18 |
| | Male | 5 | 10 | 6 |

One dog is randomly selected:

56. What is the probability that the owner of a randomly selected dog is a female?

57. What is the probability that the owners of a randomly selected dog are a couple?

58. What is the probability that a randomly selected dog is large?

59. What is the probability that a randomly selected dog is small?

60. What is the probability of randomly selecting a small dog that is owned by a male?

61. What is the probability of randomly selecting a medium size dog that is owned by a couple?

Once we have the basic notion of probabilities, we can start applying various rules. The complement of an event is all other possible outcomes. For example the complement of winning the lottery is not winning the lottery. Likewise, the complement of not winning the lottery is winning the lottery. If we know the probability of an event, we may be able to guess at the probability of the complement. For instance if there is 10% chance of winning a lottery, then there is a 90% chance of not winning the lottery.

Complement Rule:

$P(\overline{E}) = 1 - P(E)$

$P(E) = 1 - P(\overline{E})$

$\overline{E}$ means the complement of $E$. We can also say "not E" instead of $\overline{E}$.

Examples:

1. If there is a .01 or 1% probability of winning a lottery, what is the probability of not winning the lottery?

| Steps | Reasons |
|---|---|
| $P(\overline{E}) = 1 - P(E)$ | $E$ is the probability of winning the lottery. |
| $P(\overline{E}) = 1 - .01$ | $\overline{E}$ is the probability of not winning the lottery. |
| $P(\overline{E}) = .99$ | |

There is a .99 or 99% probability of not winning the lottery.

2. There is a $\frac{1}{16}$ chance that a family with four children will have all girls. What is the probability of a family with four children having at least one boy?

Remembering that the complement of at least one is none will help in recognizing when to use the complement rule. After all if there is not at least 1 boy then they must be all girls. At least 1 boy could happen many ways, but not at least 1 boy can only happen in 1 way. So, the complement rule is quite useful.

| Steps | Reasons |
|---|---|
| $P(E) = 1 - P(\overline{E})$ | Let $E$ be the event of at least 1 boy. Not at least 1 boy is all girls. So $\overline{E}$ is the event of all four girls |
| $P(\text{at least 1 boy}) = 1 - P(\text{four girls})$ | |
| $P(\overline{E}) = 1 - \frac{1}{16}$ | |
| $P(\overline{E}) = \frac{15}{16}$ | |

There is a $\frac{15}{16}$ or $.9375$ probability of having at least one boy.

Knowing that the complement of at least one is none will help in recognizing when to use the complement rule. Often students find applying the complement rule to be common sense. If there is a 20% chance of rain, then there is an 80% chance of it not raining, which is $1 - .2 = .8$ or $80\%$.

<u>Addition Rule:</u>
Look for:
1 item is selected; the word “or” is used; are the events mutually exclusive

Formula:
$\text{P(A or B)} = \text{P(A)} + \text{P(B)}$ for mutually exclusive events A and B
$\text{P(A or B)} = \text{P(A)} + \text{P(B)} - \text{P(A and B)}$ for not mutually exclusive events A and B

<u>Mutually exclusive</u> refers to events that cannot occur at the same time.

<u>Examples:</u>
3. For a-b, consider a jar with 10 tokens as follows:
5 red tokens numbered 1, 2, 3, 4, and 5
3 green tokens numbered 1, 2, and 3
2 yellow tokens numbered 1 and 2
All the tokens have the same chance of being selected. **<u>One token is randomly selected.</u>**

a. What is the probability of selecting a red or green token?

| Steps | Reasons |
|---|---|
| $\text{P(red or green)} = \text{P(red)} + \text{P(green)}$<br>$\text{P(red or green)} = \frac{5}{10} + \frac{3}{10}$<br>$\text{P(red or green)} = \frac{8}{10} \text{ or } \frac{4}{5} \text{ or } .8$ | One token is selected with “or”. Since there are no tokens that are both red and green the events are mutually exclusive. Use the addition rule for mutually exclusive. |

b. What is the probability of selecting a red or number 2 token?

| Steps | Reasons |
|---|---|
| $\text{P(red or number 2)} = \text{P(red)} + \text{P(number 2)} - \text{P(red and number 2)}$ | |
| $\text{P(red or \#2)} = \frac{5}{10} + \frac{3}{10} - \frac{1}{10}$<br>$\text{P(red or \#2)} = \frac{7}{10} \text{ or } .7$ | One token is selected with “or”. Since there is a token that is both red and number 2 the events are not mutually exclusive. Use the addition rule for not mutually exclusive. |

Here we see the reason for being careful about mutually exclusive. When we counted all the red tokens, we included the #2, red token. When we counted all #2 tokens, we include the red, #2 token. So the red, #2 token was counted twice. By subtracting it once we adjust for the double counting that occurs with not mutually exclusive.

4. In a *College Mathematics* class of the students that complete the course: 5 students get A's, 8 students get B's, 7 students get C's, 3 students get D's and 1 student gets an F. If one student is randomly selected what is the probability that the student gets an A or a B?

| Steps | Reasons |
|---|---|
| $\text{P(A or B)} = \text{P(A)} + \text{P(B)}$<br>$\text{P(A or B)} = \frac{5}{24} + \frac{8}{24}$<br>$\text{P(A or B)} = \frac{13}{24}$ or .5417 | One student is selected with "or". Since no students earn grades of both A and B, the events are mutually exclusive. Use the addition rule for mutually exclusive. Count up all the grades to get the total number of students, which is 24. Round-off the answer to 3 or 4 decimal places. |

5. Out of 200 cookies 120 contain chocolate, 60 contain walnuts, and 30 contain chocolate and walnuts. If one cookie is randomly selected, what is the probability that it contains chocolate or walnuts?

| Steps | Reasons |
|---|---|
| $\text{P(chocolate or walnut)} = \text{P(chocolate)} + \text{P(walnut)} - \text{P(chocolate and walnut)}$ | |
| $\text{P(chocolate or walnut)} = \frac{120}{200} + \frac{60}{200} - \frac{30}{200}$ | One cookie is selected. Since there are cookies that are both chocolate and walnut the events are not mutually exclusive. Use the addition rule for not mutually exclusive. |
| $\text{P(chocolate or walnut)} = \frac{150}{200}$ or $\frac{3}{4}$ or .75 | |

Again notice that we had to subtract off the probability of selecting a cookie that was chocolate and walnut because those cookies were counted under chocolate and again under walnut.

6. For a-b, learners were classified according to motivation and performance. Motivation could be low, average, or high. Performance could be poor or good. The results are in the chart:

| | | Motivation | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Performance | Poor | 14 | 8 | 3 |
| | Good | 2 | 6 | 7 |

**<u>Whenever we have one of these chart problems the first thing we do is extend the chart by including totals.</u>**

| | | Motivation | | | |
|---|---|---|---|---|---|
| | | Low | Average | High | Totals |
| Performance | Poor | 14 | 8 | 3 | 25 |
| | Good | 2 | 6 | 7 | 15 |
| | Totals | 16 | 14 | 10 | 40 |

a. One learner is randomly selected from this group of learners. Find the probability that the learner has low or average motivation (first two columns).

| Steps | Reasons |
|---|---|
| $\text{P(low or average)} = \text{P(low)} + \text{P(average)}$<br>$\text{P(low or average)} = \frac{16}{40} + \frac{14}{40}$<br>$\text{P(low or average)} = \frac{30}{40} \text{ or } \frac{3}{4} \text{ or } .75$ | One learner is selected with "or". Since no learners have both low and average motivation, the events are mutually exclusive. Use the addition rule for mutually exclusive. There are 24 students. |

b. One learner is randomly selected from this group of learners. Find the probability that the learner has average motivation or good performance.

| Steps | Reasons |
|---|---|
| $\text{P(average motiv. or good perf.)} = \text{P(aver. motiv.)} + \text{P(good perf.)} - \text{P(aver mot. and good perf.)}$ | |
| $\text{P(aver mot or good perf)} = \frac{14}{40} + \frac{15}{40} - \frac{6}{40}$<br>$\text{P(aver mot or good perf)} = \frac{23}{40} \text{ or } .575$ | Whenever we have chart problems, a row and column will overlap. In this problem there are learners with average motivation and good performance. The events are not mutually exclusive. Use the addition rule for not mutually exclusive. |

Looking at chart problems that use the addition rule, we need to watch for the following:

1. One item is being selected
2. The word "or" is used.

} The first two are required for the addition rule.

3. Two columns never intersect. So, two columns are mutually exclusive.
4. A row and a column always intersect. So, a row and column are not mutually exclusive.

Exercises

Calculate the following probabilities:

1. If there is a .0001 or .01% probability of winning a lottery, what is the probability of not winning the lottery?

2. If there is a .0005 or .05% probability of winning a lottery, what is the probability of not winning the lottery?

3. There is a $\frac{1}{128}$ chance that a family with seven children will have all boys. What is the probability of a family with seven children having at least one girl?

4. There is a $\frac{1}{64}$ chance that a family with six children will have all girls. What is the probability of a family with six children having at least one boy?

There are 400 active duty military on a remote base. 175 are in the Army, 100 are in the Air Force, 75 are in the Navy, and 50 are Marines. One of the active duty military members is randomly selected.

5. What is the probability that the military member is in the Navy or Army?

6. What is the probability that the military member is in the Marines or Air Force?

7. What is the probability that the military member is in the Air Force or Navy?

8. What is the probability that the military member is in the Army or Marines?

Consider a jar with 10 tokens as follows:

4 orange tokens numbered 1, 2, 3, and 4
3 blue tokens numbered 1, 2, and 3
3 red tokens numbered 1, 2, and 3

All the tokens have the same chance of being selected. One token is randomly selected.

9. What is the probability that an orange or blue token is selected?

10. What is the probability that a number 2 or number 3 token is selected?

11. What is the probability that a number 2 or blue token is selected?

12. What is the probability that a red or number 3 token is selected?

13. What is the probability that an orange or number 4 token is selected?

14. What is the probability that a blue or number 4 token is selected?

15. What is the probability that a number 1 or red token is selected?

16. What is the probability that an orange or number 3 token is selected?

17. What is the probability that a number 1 or number 2 token is selected?

Consider a jar with 15 tokens as follows:
7 yellow tokens numbered 1, 2, 3, 4, 5, 6, and 7
3 blue tokens numbered 1, 2, and 3
5 green tokens numbered 1, 2, 3, 4, and 5

18. What is the probability that a yellow or green token is selected?

19. What is the probability that a number 2 or number 5 token is selected?

20. What is the probability that a number 2 or green token is selected?

21. What is the probability that a yellow or number 3 token is selected?

22. What is the probability that a blue or number 5 token is selected?

23. What is the probability that a green or number 7 token is selected?

24. What is the probability that a number 1 or blue token is selected?

25. What is the probability that a blue or number 3 token is selected?

26. What is the probability that a number 1 or number 2 token is selected?

27. Out of 250 cookies 140 contain chocolate, 80 contain walnuts, and 40 contain chocolate and walnuts. If one cookie is randomly selected, what is the probability that it contains chocolate or walnuts? What is the probability that the cookie contains neither chocolate nor walnuts?

28. In a *College Mathematics* class of the students that complete the course: 6 students get A's, 9 students get B's, 8 students get C's, 4 students get D's and 2 students get F's. If one student is randomly selected what is the probability that the student gets an A or a B? What is the probability that the student gets neither an A nor a B?

Students are categorized by their attitude towards algebra and their ability to do statistics in the chart below. The students were asked if they like or dislike algebra and statistics ability was measured by their performance in an introductory statistics class. The results are in the chart:

| | | Statistics Ability | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Algebra | Likes | 5 | 12 | 10 |
| Attitude | Dislikes | 7 | 20 | 6 |

One student is randomly selected:

29. What is the probability that a randomly selected student has low statistics ability or high statistics ability?

30. What is the probability that a randomly selected student has low statistics ability or average statistics ability?

31. What is the probability that a randomly selected student has average statistics ability or high statistics ability?

32. What is the probability that a randomly selected student dislikes algebra or has average statistics ability?

33. What is the probability that a randomly selected student likes algebra or has high statistics ability?

34. What is the probability that a randomly selected student likes algebra or has low statistics ability?

35. What is the probability that a randomly selected student dislikes algebra or has high statistics ability?

36. What is the probability that a randomly selected student has average statistics ability or likes algebra?

37. What is the probability that a randomly selected student has average statistics ability or dislikes algebra?

Dog owners that own only one dog are categorized by the size of the dogs that they have and whether the owners are female, male, or a couple. The results are in the chart:

| | | Dog Size | | |
|---|---|---|---|---|
| | | Small | Medium | Large |
| | Couple | 15 | 14 | 9 |
| Owners | Female | 11 | 12 | 18 |
| | Male | 5 | 10 | 6 |

One dog is randomly selected:

38. What is the probability that the owner of a randomly selected dog is female or male?

39. What is the probability that the owners of a randomly selected dog are a couple or that the owner is male?

40. What is the probability that a randomly selected dog is small or large?

41. What is the probability that a randomly selected dog is small or medium?

42. What is the probability of randomly selecting a small dog or that the owner is a male?

43. What is the probability of randomly selecting a small dog or that the owner is a female?

44. What is the probability of randomly selecting a large dog or that the owner is a male?

45. What is the probability of randomly selecting a medium dog or that the owner is a male?

46. What is the probability of randomly selecting a medium dog or that the owners are a couple?

47. What is the probability of randomly selecting a small dog or that the owners are a couple?

48. What is the probability that the owners of a randomly selected dog are a couple or that the dog is small?

49. What is the probability that a randomly selected dog is medium or large?

When we select more than one item does the occurrence of the first event affect the probability of the second event? The answer is maybe. For example when playing cards if the cards are put back into the deck after they are drawn we still have the same 52 cards for later draws. However, if we are dealt five cards to play a card game, we are not replacing the cards and the later draws are affected by the previous draws because there are fewer cards. Here we are looking at the concept of independent and dependent events.

Events are independent if the occurrence of the first affect does not affect the probability of the occurrence of the second event. If we draw a card look at it, put the card back, then draw a second card the events of drawing the first and second card are independent. However, if the first card is not put back before drawing the second, then the events are dependent (not independent) because taking out a card affects the probability of drawing the second card. Events are dependent (not independent) if the occurrence of the first event does affect the probability of the second event.

Multiplication Rule:

Look for:

Two or more items are drawn.

The idea of “and”, which can be expressed as all, both, each, none, etc. For example Jack and Fred go to the mall is the same as saying they both go to the mall.

Are the events independent?

Formula:

$\mathrm{P(A\ and\ B) = P(A) \cdot P(B)}$

We need to consider whether or not the probability of the second event B occurring is changed by the occurrence of the first event A. (independent vs. dependent)

Examples:

1. For a-d, consider a jar with 10 tokens as follows:

5 red tokens numbered 1, 2, 3, 4, and 5

3 green tokens numbered 1, 2, and 3

2 yellow tokens numbered 1 and 2

All the tokens have the same chance of being selected.

Remember basic probabilities for an individual event. For example,

$$\mathrm{P(drawing\ a\ red\ token)} = \frac{\text{number of ways that red can occur}}{\text{total number of possible outcomes}} = \frac{5}{10} = .5$$

Remember that “$\mathrm{P(dawing\ a\ red\ token)}$” means “the probability of drawing a red token.”

a. Draw <u>two tokens without replacing</u> the first token before drawing the second token. What is the probability of drawing two red tokens?

| Steps | Reasons |
|---|---|
| $\text{P(A and B)} = \text{P(A)} \cdot \text{P(B)}$<br>$\text{P(first red and second red)} = \text{P(1st red)} \cdot \text{P(2nd red)}$ | Use the multiplication rule because two items are drawn and drawing two red tokens is the same as saying the first token is red <u>and</u> the second token is red. |
| $\text{P(1st red and 2nd red)} = \frac{5}{10} \cdot \frac{4}{9}$ | Notice that the probability of the second red is changed because the first token was not put back (dependent). There are one less red token for the numerator and one less token for the denominator. |
| $\text{P(1st red and 2nd red)} = \frac{20}{90} \text{ or } \frac{2}{9} \text{ or } .2222$ | Write answer as a reduced fraction or decimal number rounded-off to three or four places. |

b. Draw <u>one token, replace it, and draw a second token</u>. What is the probability of drawing two red tokens?

| Steps | Reasons |
|---|---|
| $\text{P(A and B)} = \text{P(A)} \cdot \text{P(B)}$<br>$\text{P(first red and second red)} = \text{P(1st red)} \cdot \text{P(2nd red)}$ | Use the multiplication rule because two items are drawn and drawing two red tokens is the same as saying the first token is red <u>and</u> the second token is red. |
| $\text{P(1st red and 2nd red)} = \frac{5}{10} \cdot \frac{5}{10}$ | Notice that the probability of the second red is not changed because the first token was put back (independent). There are the same number of tokens for the denominator and the numerator. |
| $\text{P(1st red and 2nd red)} = \frac{25}{100} \text{ or } \frac{1}{4} \text{ or } .25$ | Write answer as a reduced fraction or decimal number. |

Examples a. and b. show the importance of putting the token back (independent) and not putting the token back (dependent) when applying the multiplication rule.

c. Draw three tokens. Replace the token each time before drawing the next token. What is the probability of first drawing a red token, next drawing a green token and lastly drawing a yellow token?

| Steps | Reasons |
|---|---|
| $\text{P(1st red and 2nd green and 3rd yellow)} = \text{P(1st red)} \cdot \text{P(2nd green)} \cdot \text{P(3rd yellow)}$ | |
| $\text{P(1st red and 2nd green and 3rd yellow)}$ <br> $\frac{5}{10} \cdot \frac{3}{10} \cdot \frac{2}{10}$ <br> $\frac{30}{1000}$ or $\frac{3}{100}$ or .03 | Use the multiplication rule because three items are drawn with the idea of "and" used throughout. The first is red and the second is green and the third is yellow. Since the tokens are replaced the probabilities remain the same and the events are independent. |

d. Draw three tokens in a row. What is the probability that the first two are number 2's and last one is a number 1?

| Steps | Reasons |
|---|---|
| Multiplication Rule <br> $\text{P(1st is \#2 and 2nd is \#2 and 3rd is \#1)}$ | Two or more items are selected. We are using the idea of "and" to connect events: the first draw is a #2 and the second is #2 and the third is #1. |
| $\text{P(1st \#2)} \cdot \text{P(2nd \#2)} \cdot \text{P(3rd \#1)}$ <br> $\frac{3}{10} \cdot \frac{2}{9} \cdot \frac{3}{8}$ <br> $\frac{18}{720}$ or $\frac{1}{40}$ or .025 | When drawing three tokens in a row, the tokens are not replaced before each draw. So, these are dependent events. Notice that the denominators decrease for each drawn token that is left out. The numerator of the second draw decreases because there is one fewer number two token. For the third draw there are still 3 number one tokens. |

2. A six-sided die is rolled twice. What is the probability of rolling a 2 followed by an odd number?

The six equally likely outcomes for rolling the die once are 1,2,3,4,5, and 6.

| Steps | Reasons |
| --- | --- |
| $P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$ <br> $P(\text{first \#2 and second odd}) = P(\text{1st \#2}) \cdot P(\text{2nd odd})$ | Use the multiplication rule because the die is rolled twice and the first roll is a number 2 and the second roll is odd. |
| $P(\text{1st red and 2nd red}) = \frac{1}{6} \cdot \frac{3}{6}$ | Notice that the probability of the second roll is not affected by the first roll because the die is not changed in any way (independent). There is one #2 out of six possible outcomes. There are three odd possibilities (1,3,5) out of six total possible outcomes. |
| $P(\text{1st red and 2nd red}) = \frac{3}{36} \text{ or } \frac{1}{12} \text{ or } .0833$ | Write answer as a reduced fraction or decimal number. |

Conditional probabilities can be thought of as an alteration of the original problem. When we have the multiplication rule for dependent events the probability of the second event really is a conditional probability. In the example above the probability of drawing the second red ball given that the first red ball was drawn and not replaced is $\frac{4}{9}$ instead of $\frac{5}{10}$. The trick to conditional probabilities is to restrict the problem to whatever comes after the given and ignore the rest of the information. There is also a formula.

Formula for Conditional Probabilities

$$P(\text{B given A}) = \frac{\text{number of outcomes common to B and A}}{\text{number of outcomes for A}}$$

If we look at the formula carefully, we are finding the probability of event B occurring restricted to the situation where the event A has occurred.

Rather than trying to apply the formula, the easiest way to do conditional probabilities will be to just look at the outcomes after the word “given”.

Examples:

3. For a-c, consider our jar with 10 tokens as follows:
    - 5 red tokens numbered 1, 2, 3, 4, and 5
    - 3 green tokens numbered 1, 2, and 3
    - 2 yellow tokens numbered 1 and 2

All the tokens have the same chance of being selected.

a. On one draw what is the probability of drawing a number 2 token given that a green token is drawn?

$$\text{P}(\#2 \text{ drawn given a green token drawn})$$

Since it is given that a green token is drawn we are only interested in the green tokens.
There are three green tokens numbered 1, 2, and 3.
There is one number 2 out of three green tokens. So,

$$\text{P}(\#2 \text{ drawn given a green token drawn}) = \frac{1}{3} \text{ or } .3333$$

b. On one draw what is the probability of drawing an odd number given that a red token is drawn?

$$\text{P}(\text{odd drawn given a red token is drawn})$$

Since it is given that a red token is drawn we are only interested in the red tokens.
There are five red tokens numbered 1, 2, 3, 4, and 5.
There are three odd red tokens (1,3,5) out of five red tokens. So,

$$\text{P}(\text{odd drawn given a red token is drawn}) = \frac{3}{5} \text{ or } .6$$

c. On one draw what the probability of drawing a green token given that a number 3 token is drawn.

$$\text{P}(\text{green token given number 3 token is drawn})$$

Since it is given that a number 3 token is drawn we are only interested in the number 3 tokens.
There are two #3 tokens: red #3 and green #3. So,

$$\text{P}(\text{green token given number 3 token is drawn}) = \frac{1}{2} \text{ or } .5$$

4. As before for a-b, learners were classified according to motivation and performance. Motivation could be low, average, or high. Performance could be poor or good. The results are in the chart:

| | | Motivation | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Performance | Poor | 14 | 8 | 3 |
| | Good | 2 | 6 | 7 |

**<u>Whenever we have one of these chart problems the first thing we do is extend the chart by including totals.</u>**

| | | Motivation | | | |
|---|---|---|---|---|---|
| | | Low | Average | High | Totals |
| Performance | Poor | 14 | 8 | 3 | 25 |
| | Good | 2 | 6 | 7 | 15 |
| | Totals | 16 | 14 | 10 | 40 |

a. One learner is randomly selected from this group of learners. Find the probability that the learner has low motivation given that he has good performance.

When we calculate conditional probabilities from a chart, we are only looking at a single row or a single column. When we say:
"probability that the learner has low motivation given that he has good performance"

The phrase after "given" tells us where to look. So, for good performance only (it is given that there is good performance), we look at the good performance row.

| | | Motivation | | | |
|---|---|---|---|---|---|
| | | Low | Average | High | Totals |
| Performance | Poor | | | | |
| | Good | 2 | 6 | 7 | 15 |
| | Totals | | | | |

The probability of low motivation for the above row of good performance is $\frac{2}{15}$ or .1333

We can also use the formula for conditional probabilities with the chart:

$$\text{P(B given A)} = \frac{\text{number of outcomes common to B and A}}{\text{number of outcomes for A}}$$

$$\text{P(low motiv. given good performance)} = \frac{\text{number of good performance and low motivation}}{\text{number of outcomes for good performance}} = \frac{2}{15} \text{ or } .1333$$

b. One learner is randomly selected from this group of learners. Find the probability that the learner has poor performance given that he has average motivation.

Since we are given average motivation, we need look at only the average motivation column.

| | | Motivation | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Performance | Poor | | 8 | |
| | Good | | 6 | |
| | Totals | | 14 | |

The probability of poor performance for the above column of average motivation is $\frac{8}{14}$ or $\frac{4}{7}$ or $.5714$

We can also use the formula for conditional probabilities with the original chart:

$$\text{P(B given A)} = \frac{\text{number of outcomes common to B and A}}{\text{number of outcomes for A}}$$

$$\text{P(poor perform. given average motiv.)} = \frac{\text{number of poor perform and average motivation}}{\text{number of outcomes for average motivation}}$$

$$= \frac{8}{14} \text{ or } \frac{4}{7} \text{ or } .5714$$

5. There are three switches each with a 20% chance of failing.

a. What is the probability that all three switches fail?

| Steps | Reasons |
|---|---|
| $\text{P(three switches fail)} = \text{P(1st fail and 2nd fails and 3rd fails)}$ | All three is the same as “and.” |
| $\text{P(1st fails)} \cdot \text{P(2nd fails)} \cdot \text{P(3rd fails)}$ | Multiplication rule with three items selected with “and.” |
| $.2 \cdot .2 \cdot .2$ | |
| $.008$ | The probability of each switch failing is .2 and then multiply. |

b. What is the probability that a single switch does not fail?

| Steps | Reasons |
|---|---|
| $\text{P(a single switch does not fail)}$ | “does not fail” is the complement of “does fail” |
| $\text{P(does not fail)} = 1 - \text{P(fails)}$ | Complement rule |
| $\text{P(does not fail)} = 1 - .2 = .8$ | |

c. Of the three switches what is the probability that no switches fail?

| Steps | Reasons |
|---|---|
| $\text{P(no switches fail )} = \text{P(1st not fail and 2nd not fail and 3rd not fail)}$ | No switches is the same as "and." |
| $\text{P(1st not fail)} \cdot \text{P(2nd not fail)} \cdot \text{P(3rd not fail)}$ | Multiplication rule with three items selected with "and." |
| $.8 \cdot .8 \cdot .8$ | The probability of each switch not failing is .8 and then we multiply. |
| $.512$ | |

d. What is the probability that at least one switch does not fail?

| Steps | Reasons |
|---|---|
| $\text{P(at least one does not fail )}$ | "at least one" can occur many ways: any one switch could fail, any two switches could fail, or any three switches could fail. In fact there are so many possibilities that it will be easier to use the compliment rule |
| $\text{P(at least one fails)} = 1 - \text{P(not at least one fails)}$ | Complement rule |
| $\text{P(at least one fails)} = 1 - \text{P(none fail)}$ | "not at least one" is the same as saying "none" That is the real trick. If we see "at least one", we may want to use the compliment rule. |
| $1 - .512$ | The probability that none of the three switches fails is coming directly from step c. of this problem. |
| $.488$ | |

Exercises

Calculate the following:

A six-sided die with equally likely outcomes {1,2,3,4,5,6} is rolled twice. Find the probability of

1. First rolling an even number and then rolling an odd number.

2. First rolling a number 3 and then rolling an even number.

3. First rolling a number greater than 2 and then rolling an even number.

4. First rolling an odd number and then rolling a number 5.

5. First rolling a number that is at least 3 and then rolling a number that is at most 3.

6. First rolling a number less than 5 and then rolling a number more than 4

A six-sided die with equally likely outcomes {1,2,3,4,5,6} is rolled three times. Find the probability of

7. Rolling three even numbers.

8. Rolling three odd numbers.

9. Rolling an even number on the first roll, rolling at least a number 5 on the second roll, and rolling a number more than 2 on the third roll.

10. Rolling three numbers that are all at most 5.

11. Rolling three numbers that are all at least 3.

12. Rolling an odd number on the first roll, rolling at least a number 4 on the second roll, and rolling a number more than 2 on the third roll.

Consider a jar with 10 tokens as follows:

4 orange tokens numbered 1, 2, 3, and 4
3 blue tokens numbered 1, 2, and 3
3 red tokens numbered 1, 2, and 3

All the tokens have the same chance of being selected.

13. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are orange?

14. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are blue?

15. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are orange?

16. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are blue?

17. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 2 and the second token is a number 3?

18. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 1 and the second token is a number 4?

19. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is a number 3 and the second token is number 4?

20. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is number 1 and the second token is number 2?

21. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are blue?

22. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are orange?

23. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are blue?

24. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are orange?

25. One token is drawn. What is the probability that the token is a number 1 given that it is orange?

26. One token is drawn. What is the probability that the token is a number 2 given that it is blue?

27. One token is drawn. What is the probability that the token is a number 3 given that it is red?

28. One token is drawn. What is the probability that the token is an even number given that it is orange?

29. There are three switches each with a 30% chance of failing.
    a. What is the probability that all three switches fail?
    b. What is the probability that a single switch does not fail?
    c. Of the three switches what is the probability that no switches fail?
    d. Of the three switches what is the probability that at least one switch fails?

30. There are three switches each with a 25% chance of failing.
    a. What is the probability that all three switches fail?
    b. What is the probability that a single switch does not fail?
    c. Of the three switches what is the probability that no switches fail?
    d. Of the three switches what is the probability that at least one switch fails?

Consider a jar with 15 tokens as follows:

7 yellow tokens numbered 1, 2, 3, 4, 5, 6, and 7
3 blue tokens numbered 1, 2, and 3
5 green tokens numbered 1, 2, 3, 4, and 5

All the tokens have the same chance of being selected.

31. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are yellow?

32. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are green?

33. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are yellow?

34. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are green?

35. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 2 and the second token is a number 3?

36. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 2 and the second token is a number 5?

37. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is a number 3 and the second token is a number 6?

38. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is a number 1 and the second token is a number 2?

39. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are yellow?

40. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are green?

41. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are yellow?

42. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are green?

43. One token is drawn. What is the probability that the token is a number 1 given that it is yellow?

44. One token is drawn. What is the probability that the token is a number 2 given that it is blue?

45. One token is drawn. What is the probability that the token is a number 3 given that it is green?

46. One token is drawn. What is the probability that the token is an odd number given that it is yellow?

47. One token is drawn. What is the probability that the token is an even number given that it is yellow?

48. One token is drawn. What is the probability that the token is an odd number given that it is blue?

Students are categorized by their attitude towards algebra and their ability to do statistics in the chart below. The students were asked if they like or dislike algebra and statistics ability was measured by their performance in an introductory statistics class. The results are in the chart:

| | | Statistics Ability | | |
|---|---|---|---|---|
| | | Low | Average | High |
| Algebra | Likes | 5 | 12 | 10 |
| Attitude | Dislikes | 7 | 20 | 6 |

One student is randomly selected:

49. What is the probability that a randomly selected student has low statistics ability?

50. What is the probability that a randomly selected student has high statistics ability?

51. What is the probability that a randomly selected student has average statistics ability given that the student likes algebra?

52. What is the probability that a randomly selected student has average statistics ability given that the student dislikes algebra?

53. What is the probability that a randomly selected student likes algebra and has high statistics ability?

54. What is the probability that a randomly selected student likes algebra and has low statistics ability?

55. What is the probability that a randomly selected student likes algebra or has high statistics ability?

56. What is the probability that a randomly selected student dislikes algebra or has low statistics ability?

57. What is the probability that a randomly selected student likes algebra given that the student has average statistics ability?

58. What is the probability that a randomly selected student dislikes algebra given that the student has average statistics ability?

Dog owners that own only one dog are categorized by the size of the dogs that they have and whether the owners are female, male, or a couple. The results are in the chart:

| | | Dog Size | | |
|---|---|---|---|---|
| | | Small | Medium | Large |
| | Couple | 15 | 14 | 9 |
| Owners | Female | 11 | 12 | 18 |
| | Male | 5 | 10 | 6 |

One dog is randomly selected:

59. What is the probability that the owner of a randomly selected dog is a male?

60. What is the probability that a randomly selected dog is medium?

61. What is the probability of randomly selecting a medium dog that is owned by a male?

62. What is the probability of randomly selecting a large dog that is owned by a couple?

63. What is the probability of randomly selecting a medium dog given that is owned by a male?

64. What is the probability of randomly selecting a large dog given that is owned by a couple?

65. What is the probability of randomly selecting a small dog given that the owner is male?

66. What is the probability of randomly selecting a small dog given that the owner is female?

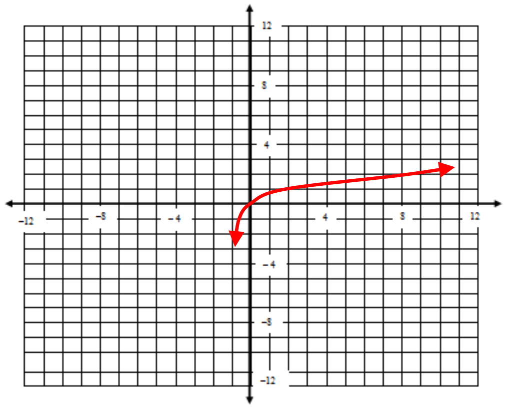
67. What is the probability of randomly selecting a medium dog or that the owner is a female?

68. What is the probability of randomly selecting a large dog or that the owner is a female?

69. What is the probability that the owners of a randomly selected dog are a couple given that the dog is medium?

70. What is the probability that the owner of a randomly selected dog is a male given that the dog is small?

19. $y = \log_3(x+1)$




21. 4602 people will hear about the product after $40,000 is spent on advertising. 6176 people will hear about the product after $1,500,000 is spent on advertising. It does not appear that the extra advertising dollars will be well spent.

23. The growth rate is 9.9%, which is very high for inflation.

## Exercise Set 6.1

1. $2080

3. $225,000

5. $162

7. $580.83

9. $184.93

11. $16,695.21

13. $25,920

15. $15,750

17. $14,978

19. 6.5%

21. 7.6%

23. $12,791

25. $8733

27. $69.35

29. $25,813.70

31. 3.7%

33. $48,781

Exercise Set 6.2
For future value problems, round to the nearest dollar.
1. $26,044

3. $557,265

5. $333,184

7. $25,871

9. $295,967

11. $207,484 ; If this were simple interest instead of annually compounded interest the future value would be $136,800. The future value for annually compounded interest is more than simple interest because the interest that is earned and applied each year grows for the rest of the investment time. For simple interest the interest is applied once at the end of the investment time based on the original principal only.

13. $125,225

15. $73,288

17. $1,076,891

19. $50,186

21. $8,723

23. $63,510

25. $63

For present value problems, round-up to the nearest dollar.

27. $24,263

29. $60,570

31. $35,801

33. $52,237

35. $11,655

37. $12,690

39. $349,460

41. a. $203,849
    b. $552,967
    c. $91,961 at 7% ; $452,469 at 3%
    d. The twenty-five year old could invest some money every month.

43. $39,445.74

Exercise Set 6.3

1. $158,853 ; $68,853

3. $1,489,153 ; $1,201,153

5. a. $603,518 ; $387,518
   b. $349,312 ; $133,312
   c. By starting to save for retirement earlier, the young person earns $254,206 extra in interest for the same amount of money put in.

7. $662 ; $721,960

9. $3900 ; $298,000

11. a. $558
    b. $1453
    c. It is important for the couple to start early because they pay less.
    d. $363 ; $752

13. $376,249 ; $229,249

15. $403,866 ; deposits account for $16,154,640; interest accounts for $3,845,360

Exercise Set 6.4

For the following questions, round to the nearest dollar.

1. a. $13,750
   b. $261,250
   c. $3918.75
   d. $1443
   e. $258,230

3. a. $174,000
   b. $406,000
   c. $8120
   d. $3054
   e. $143,720

5. a. $1802 ; $339,970
   b. $2615 ; $70,050
   c. If the couple can afford the down payment and the monthly payments, the second option will be much better because it saves them $269,920 in interest.

7. a. $1277 ; $387,960
   b. $1471 ; $84,780
   c. If the bachelor can afford the down payment and the monthly payments, the second option will be much better because it saves the bachelor $303,180 in interest.

9. $431 ; $17,720

11. $872 ; $1392

13. $776 ; $33,120

Exercise Set 7.1

1. 24

3. 1

5. 182

7. 639,200

9. $28$

11. $\approx 8.04531 \text{ x } 10^{13}$

13. $67{,}600{,}000$ ; Two possible license plates are AA 12345 and BZ 73912 .

15. $45$ ; Two possible breakfast options are coffee/omelet/booth and juice/pancakes/counter .

17. ${}_{10}P_2 = 90$

19. ${}_{20}P_4 = 116{,}280$

21. ${}_{18}C_4 = 3060$

23. ${}_{25}C_6 = 177{,}100$

25. ${}_{10}C_2 \cdot {}_8C_5 = 2520$

27. $36$

29. $32$

31. ${}_{15}P_4 = 32{,}760$

33. ${}_{30}P_4 = 657{,}720$

35. ${}_{200}P_3 = 7{,}880{,}400$

37. ${}_{150}C_3 = 551{,}300$

39. ${}_8C_3 \cdot {}_{15}C_5 = 168{,}168$

41. $96$

43. $20 \cdot 20 \cdot 20 = 8000$ ways if the flavors can be repeated. ${}_{20}P_3 = 6840$ ways if the flavors must be different.

45. ${}_{100}C_{15} \approx 2.53338 \text{ x } 10^{17}$

47. ${}_{80}P_2 = 6320$

49. ${}_{15}C_2 \cdot {}_9C_3 = 8820$

Exercise Set 7.2

1. $\frac{3}{6}$ or .5

3. $\frac{2}{3}$ or .667

5. $\frac{1}{3}$ or .333

7. $\frac{0}{6}$ or 0

9. $\frac{3}{8}$ or .375

11. $\frac{1}{2}$ or .5

13. $\frac{1}{8}$ or .125

15. $\frac{7}{8}$ or .875

17. $\frac{0}{8}$ or 0

19. $\frac{3}{8}$ or .375

21. $\frac{11}{16}$ or .6875

23. $\frac{1}{4}$ or .25

25. $\frac{5}{16}$ or .3125

27. $\frac{11}{16}$ or .6875

29. $\frac{0}{16}$ or 0

31. $\frac{3}{10}$ or .3

33. $\frac{3}{10}$ or .3

35. $\frac{7}{15}$ or .467

37. $\frac{3}{5}$ or .6

39. $\frac{1}{5}$ or $.2$

41. $\frac{1}{2}$ or $.5$

43. $\frac{7}{12}$ or $.583$

45. $\frac{5}{18}$ or $.278$

47. $\frac{0}{36}$ or $0$

49. $\frac{11}{20}$ or $.55$

51. $\frac{9}{20}$ or $.45$

53. $\frac{1}{6}$ or $.167$

55. $\frac{1}{5}$ or $.2$

57. $\frac{19}{50}$ or $.38$

59. $\frac{31}{100}$ or $.31$

61. $\frac{7}{50}$ or $.14$

Exercise Set 7.3

1. $.9999$ or $99.99\%$

3. $\frac{127}{128}$ or $.992$

5. $\frac{5}{8}$ or $.625$

7. $\frac{7}{16}$ or $.4375$

9. $\frac{7}{10}$ or $.7$

11. $\frac{1}{2}$ or $.5$

13. $\frac{2}{5}$ or .4

15. $\frac{1}{2}$ or .5

17. $\frac{3}{5}$ or .6

19. $\frac{1}{3}$ or .333

21. $\frac{3}{5}$ or .6

23. $\frac{2}{5}$ or .4

25. $\frac{1}{3}$ or .333

27. $\mathrm{P(chocolate\ or\ walnut)} = \frac{18}{25}$ or .72 $\mathrm{P(neither\ chocolate\ nor\ walnut)} = \frac{7}{25}$ or .28

29. $\frac{7}{15}$ or .467

31. $\frac{4}{5}$ or .8

33. $\frac{11}{20}$ or .55

35. $\frac{43}{60}$ or .717

37. $\frac{3}{4}$ or .75

39. $\frac{59}{100}$ or .59

41. $\frac{67}{100}$ or .67

43. $\frac{61}{100}$ or .61

45. $\frac{47}{100}$ or .47

47. $\frac{27}{50}$ or .54

49. $\frac{69}{100}$ or .69

Exercise Set 7.4

1. $\frac{1}{4}$ or .25

3. $\frac{1}{3}$ or .333

5. $\frac{1}{3}$ or .333

7. $\frac{1}{8}$ or .125

9. $\frac{1}{9}$ or .111

11. $\frac{8}{27}$ or .296

13. $\frac{4}{25}$ or .16

15. $\frac{2}{15}$ or .133

17. $\frac{9}{100}$ or .09

19. $\frac{1}{30}$ or .0333

21. $\frac{27}{1000}$ or .027

23. $\frac{1}{120}$ or .00833

25. $\frac{1}{4}$ or .25

27. $\frac{1}{3}$ or .333

29. a. 0.027
    b. 0.7
    c. 0.343
    d. 0.657

31. $\frac{49}{225}$ or .217

33. $\frac{1}{5}$ or .2

35. $\frac{1}{25}$ or .04

37. $\frac{1}{70}$ or .0143

39. $\frac{343}{3375}$ or .102

41. $\frac{1}{13}$ or .0769

43. $\frac{1}{7}$ or .143

45. $\frac{1}{5}$ or .2

47. $\frac{3}{7}$ or .429

49. $\frac{1}{5}$ or .2

51. $\frac{4}{9}$ or .444

53. $\frac{1}{6}$ or .167

55. $\frac{11}{20}$ or .55

57. $\frac{3}{8}$ or .375

59. $\frac{21}{100}$ or .21

61. $\frac{1}{10}$ or .1

63. $\frac{10}{21}$ or .476

65. $\frac{5}{21}$ or .238

67. $\frac{13}{20}$ or .65

69. $\frac{7}{18}$ or .389

# Math 103 Formula Sheet

## Financial Management

Simple Interest: $\text{Int} = \text{Prt}$

Future Value for Simple Interest: $FV = P(1 + rt)$

Future Value for Compound Interest: $FV = P\left(1 + \frac{r}{n}\right)^{nt}$

Present Value for Compound Interest: $P = \dfrac{FV}{\left(1 + \frac{r}{n}\right)^{nt}}$

Future Value for continuous compounding: $FV = Pe^{r \cdot t}$

Effective Annual Yield: $EAY = \left(1 + \frac{r}{n}\right)^{n} - 1$

Future Value of an Annuity (Pmt is the amount of each deposit): $FV = \dfrac{Pmt\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)}$

Periodic deposits for an Annuity (FV is the future value of the annuity): $Pmt = \dfrac{FV\left(\frac{r}{n}\right)}{\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}$

Periodic Mortgage Payments (B is the amount of mortgage): $Pmt = \dfrac{B\left(\frac{r}{n}\right)}{\left[1 - \left(1 + \frac{r}{n}\right)^{-nt}\right]}$

## Probability and Counting Rules

Permutation rule: ${}_{n}P_{k} = \dfrac{n!}{(n-k)!}$

Combination rule: ${}_{n}C_{k} = \dfrac{n!}{(n-k)!\,k!}$

$P(\overline{E}) = 1 - P(E)$

$P(E) = 1 - P(\overline{E})$

$P(A \text{ or } B) = P(A) + P(B)$

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$P(A \text{ and } B) = P(A) \cdot P(B)$

$P(B \text{ given } A) = \dfrac{\text{number of common outcomes for B and A}}{\text{number of outcomes within A}}$

## Statistics

Mean for individual data: $\bar{x} = \dfrac{\sum x}{n}$

Mean for grouped data: $\bar{x} = \dfrac{\sum f \cdot x_m}{n}$

Standard Deviation: $s = \sqrt{\dfrac{\sum (x - \bar{x})^2}{n - 1}}$

Z-score: $z = \dfrac{x - \bar{x}}{s}$

$\bar{x}$ = mean    x= data values    $\sum$ = add all the values    f = frequency    $x_m$= class or class midpoint    s = standard deviation