4

We are working with degree two or quadratic expressions ($ax^2 + bx + c$) and equations ($ax^2 + bx + c = 0$). We see techniques such as multiplying and factoring expressions and solving equations using factoring or the quadratic formula.

Course Outcomes:

- Demonstrate mastery of algebraic skills

4.1 Multiplying Algebraic Expressions and FOIL

Multiplication of variables with exponents is reviewed. A monomial is multiplied by a polynomial using the distributive property. Binomials are multiplied by binomials using FOIL (First, Outer, Inner, Last)

4.2 Factoring and Solving by Factoring

Students learn to factor out a common factor and factor trinomials in the forms of $x^2 + bx + c$ or $ax^2 + bx + c$. Solving by factoring and some basic translation problems for degree two equations are covered.

4.3 Quadratic Formula

Students learn to solve degree two equations by factoring and applying the quadratic formula. Quadratic equations may or may not be able to be solved by factoring.

When multiplying monomials and polynomials we add exponents using the following rule:

$x^m \cdot x^n = x^{m+n}$

All we are saying is that $x^2 \cdot x^4 = (x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^6$. It is easier to add the 2 and 4 to get the 6 rather than writing it out.

Monomial times a polynomial:
Multiply each term in the polynomial by the monomial by multiplying the numbers and adding the exponents.

Examples

1. Multiply: $5x(2x^2 - x + 3)$

Steps	Reasons
$5x(2x^2 - x + 3)$	Use the distributive property.
$5x(2x^2 - x + 3)$	Multiply each term in the polynomial by $5x$: 1. Multiply numbers (coefficients). 2. Multiply each variable by adding exponents, where x has exponent of 1.
$(5x)(2x^2) - (5x)(x) + (5x)(3)$	Here the distributive property is written out. We can really leave this step out by thinking of arrows.
$10x^3 - 5x^2 + 15x$	

2. Multiply: $3x^2y^3(2x^4 - xy^6 + 5x^5)$

Steps	Reasons
$3x^2y^3(2x^4 - xy^6 + 5x^5)$	Use the distributive property.
$3x^2y^3(2x^4 - xy^6 + 5x^5)$	Multiply each term in the polynomial by $3x^2y^3$: 1. Multiply numbers (coefficients). 2. Multiply each variable by adding exponents.
$(3x^2y^3)(2x^4) - (3x^2y^3)(xy^6) + (3x^2y^3)(5x^5)$	Here the distributive property is written out. We can really leave this step out by thinking of arrows.
$6x^6y^3 - 3x^3y^9 + 15x^7y^3$	Write the answer.

3. Multiply: $-2y^2\left(3y^5 - 4y^4 - y^3\right)$

Steps	Reasons
$-2y^2\left(3y^5 - 4y^4 - y^3\right)$	Use the distributive property.
$\left(-2y^2\right)\left(3y^5\right)+\left(2y^2\right)\left(4y^4\right)+\left(2y^2\right)\left(y^3\right)$	Multiply $\left(-2y^2\right)$by each term. At least take care of your negative signs and subtraction. A negative number multiplied by a number being subtracted becomes addition. With practice, this step may be skipped by drawing arrows to indicate the correct multiplication.
$-6y^7 + 8y^6 + 2y^5$	Multiply the numbers. Multiply the variables by adding their exponents.

The FOIL method for multiplication of binomials is very important in algebra. FOIL is an acronym for:

First: Multiply the first terms of each binomial.
Outer: Multiply the outer terms of the binomials.
Inner: Multiply the inner terms of the binomials.
Last: Multiply the last terms of the binomials.

Examples

4. Multiply: $(x+3)(x+5)$

Steps	Reasons
$(x+3)(x+5)$	To multiply two binomials use FOIL
x^2	First: Multiply the first terms of each binomial.
5x	Outer: Multiply the outer terms of the binomials.
3x	Inner: Multiply the inner terms of the binomials.
15	Last: Multiply the last terms of the binomials.
$x^2 + 3x + 5x + 15$	Put all the terms together.
$x^2 + 8x + 15$	Collect like terms.

This problem should be done in three steps since the FOIL part can be done mentally.

5. Multiply: $(3x-7)(2x+5)$

Steps	Reasons
$(3x-7)(2x+5)$	Use FOIL to do all the multiplications mentally. You can draw arrows between the first, outer, inner, last terms to help focus on the multiplication.
$6x^2+15x-14x-35$	First: $3x \cdot 2x = 6x^2$ Outer: $3x \cdot 5 = 15x$ Inner: $-7 \cdot 2x = -14x$ Last: $-7 \cdot 5 = -35$
$6x^2+x-35$	Collect like terms.

Students may wonder where FOIL comes from. Really we are using the distributive property twice. If we do example 4 using the distributive property we will get the same answer.

Steps	Reasons
$(x+3)(x+5)$	
	Distribute the whole expression $(x+3)$
$(x+3)x+(x+3)5$	
	Distribute the x and the 5.
$x \cdot x+3x+5x+15$	
	We end up with the same operations and answers as if we did FOIL
$x^2+8x+15$	

It is easier to multiply these binomials by FOIL. So, we generally do these problems using FOIL.

Exercises

Multiply:

1. $5x(2x^2 + 3x - 4)$
2. $2x(3x^2 - 4x + 5)$
3. $7x(3x^2 - 5x - 11)$
4. $4x(3x^2 - 2x - 9)$
5. $-3x^2(2x^2 - 3x - 6)$
6. $-7x^2(3x^2 - 5x - 9)$
7. $-4x^2(2x^2 - 5x - 6)$
8. $-6x^2(9x^2 - 2x + 5)$
9. $3x^4(5x^4 - 3x^3 - 2x^2)$
10. $7x^5(9x^5 - 3x^4 - 7x^3)$
11. $4xy(3x^2 - 2xy + 5y^2)$
12. $7xy(3y^2 - 5xy - 11y^2)$
13. $-3x^2y(15xy^2 - 10x^2y - 8x^2y^2)$
14. $-7xy^2(12x^2y^2 - 9x^2y + 6xy^2)$
15. $8x^2y(5x^2y - 10xy - 7xy^2 - 12x^2y^2)$
16. $5xy^2(4x^2 - 6xy^2 - 8x^2y - 9x^2y^2)$
17. $(3x + 2)(5x + 4)$
18. $(2x + 1)(3x + 4)$

19. $(5x - 2)(3x + 7)$

20. $(3x + 8)(2x - 9)$

21. $(7x - 2)(5x - 11)$

22. $(4x + 2)(3x + 8)$

23. $(7x - 3)(6x - 5)$

24. $(3x - 11)(2x - 9)$

25. $(11x - 6)(5x - 4)$

26. $(10x - 7)(8x - 6)$

27. $(9x + 6)(8x - 7)$

28. $(8x - 7)(9x + 6)$

29. $(5x - 10)(3x + 12)$

30. $(7x - 15)(4x - 10)$

31. $(3x^2 - 9)(2x^2 + 4)$

32. $(5x^2 - 12)(3x^2 - 15)$

33. $(8x^2 - 14)(9x^2 - 10)$

34. $(7x^2 + 8)(12x^2 - 9)$

To factor is to write an expression as the product of its factors. The number 6 can be factored into $2 \cdot 3$

Factoring out a common factor is based on doing the distributive property in reverse.

$$a \cdot b + a \cdot c = a(b + c)$$

Look for the common factor in all terms:

1. Look for a number that is a common factor.
2. Look for the variables that appear in all terms and use the smallest exponent.

Examples

1. Factor: $15x^5 + 25x^4 - 10x^3$

Steps	Reasons
$15x^5 + 25x^4 - 10x^3$	5 is a factor of all three terms x^3 is a factor of all three terms. (Use the smallest exponent.)
$5x^3(3x^2 + 5x - 2)$	Divide each term by the $5x^3$. Actually, I think of what I need to multiply $5x^3$ by to get each of the original terms of the $15x^5 + 25x^4 - 10x^3$

You can check by multiplying:

$$5x^3(3x^2 + 5x - 2) = 15x^5 + 25x^4 - 10x^3$$

2. Factor: $12x^2y^2 - 8x^2y - 40x^2$

Steps	Reasons
$12x^2y^2 - 8x^2y - 40x^2$	4 is a factor of all three terms x^2 is a factor of all three terms.
$4x^2(3y^2 - 2y - 10)$	Divide each term by the $4x^2$. I think of what I need to multiply $4x^2$ by to get each of the original terms of the $12x^2y^2 - 8x^2y - 40x^2$

You can check by multiplying.

Factoring polynomials of the form $x^2 + bx + c$

factors of "c" whose sum is "b"
write those numbers (x …)(x…)

After we do an example we can see why it works by checking.

3. Factor x^2+5x+6

Steps	Reasons
x^2+5x+6	b is 5 and c is 6
	Look for the factors of "c" whose sum is "b".
	List factors of 6
$(x+2)(x+3)$	1·6, -1·(-6), 2·3, -2·(-3)
	2 and 3 have a sum of 5
	Write those numbers after the (x …)(x …)

When we check by multiplying, we see why the sum of the numbers is 5 and the product is 6.

$(x+2)(x+3)$	In the FOIL we will always get x·x
$x \cdot x+3x+2x+6$	2x + 3x = 5x We needed the sum to be 5
	2·3 = 6 We only tried factors of 6
x^2+5x+6	

4. Factor: $x^2-5x-24$

Steps	Reasons
$x^2-5x-24$	b is –5 and c is –24
List factors of -24	
	Look for the factors of "c" whose sum is "b".
–2·12 , –3·8, 3·(–8)	
3 and –8 have a sum of –24	
$(x-8)(x+3)$	Write those numbers after the (x …)(x …)

5. Factor: x^2+3x+7

Steps	Reasons
x^2+3x+7	b is 3 and c is 7
List factors of 7	
	Look for the factors of "c" whose sum is "b".
1·7, –1·(–7)	None of the factors of 7 add up to 3

So, x^2+3x+7 does not factor. We can also say that x^2+3x+7 is prime.

6. Factor: $6x^3 - 12x^2 - 18x$

Steps	Reasons
$6x^3 - 12x^2 - 18x$	First factor out the common factor.
$6x(x^2 - 2x - 3)$	
$6x(x-3)(x+1)$	The factors of –3 whose sum is –2 are – 3 and 1. Keep the common factor of 6x.

Factoring polynomials of the form $ax^2 + bx + c$ by Guess and Check:

We are doing FOIL backwards. ax^2 is coming from the first terms and c is coming from the last terms. We can check all possibilities that will give us **ax^2** and **c**. Then we check the FOIL to see if we have the right **bx** term.

Examples

7. Factor $2x^2 + 7x + 3$

Steps	Reasons
$(2x \quad)(x \quad)$	$2x^2$ must factor into 2x and x in the first two terms.
Check: $(2x+3)(x+1)$ $(2x+1)(x+3)$ $(2x-1)(x-3)$ $(2x-3)(x-1)$	The factors of 3 are 1,3 and . Try them in both possible orders. They would not be negative because the middle term 7x is positive.
$(2x+1)(x+3)$	If you check the FOIL the answer is the second choice.

Check:
$(2x+1)(x+3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$

Here we are guessing 2x and x for the first part of the factors because they will give us $2x^2$ in the FOIL. We guess 1 and 3 for the second part of the factors because they will give us 3 in the FOIL. We have to try the various combinations and do the FOIL to see which combination works.

8. Factor: $6x^2 - 23x - 4$

Steps	Reasons
$(3x \quad)(2x \quad)$ *or* $(6x \quad)(x \quad)$	$6x^2$ must factor into 3x and 2x or 6x and x in the first two terms.
$(3x-4)(2x+1)$ $(3x+4)(2x-1)$ $(3x-1)(2x+4)$ $(3x+1)(2x-4)$ $(3x-2)(2x+2)$ $(3x+2)(2x-2)$ $(6x-4)(x+1)$ $(6x+4)(x-1)$ $(6x-1)(x+4)$ $(6x+1)(x-4)$ $(6x-2)(x+2)$ $(6x+2)(x-2)$	The second part of the two factors has -4,1 or 4,-1, or 2,-2 in either order. This leaves us with 12 possibilities to check. Being organized helps but this method works best when the “a” and “c” are prime numbers like in the first example.
$(6x+1)(x-4)$	If you check the FOIL for all of the combinations, it turns out to be the second to the last.

Types of factoring:

1. Factor out any common factor
    - Always try this first.
    - This is the distributive property backwards.

2. Check the form of the polynomial:

    $x^2 + bx + c$
    - factors of "c" whose sum is "b"
    - write those numbers (x …)(x…)

    $ax^2 + bx + c$
    - We are doing FOIL backwards. ax^2 is coming from the first terms and c is coming from the last terms. We can check all possibilities that will give us **ax^2** and **c**. Then we check the FOIL to see if we have the right **bx** term.
    - write those numbers (_x …)(_x…)

yields ax^2 results in c

Now that we can factor, we can solve some equations that have exponents for the variable. If two numbers are multiplied together to get a product of zero, then one of the must be zero. In other words, If $a \cdot b = 0$, then either a is zero or b is zero.

Example

9. Solve $(x-3)(x+2)=0$

Steps	Reasons
$(x-3)(x+2)=0$	If the product equals zero, then one of the factors must equal zero.
$x-3=0 \ or \ x+2=0$	Set each factor equal to zero.
$x=3 \quad or \ x=-2$	Solve the two resulting equations.

A quadratic equation has the form ax^2+bx+c where a,b,c are real numbers and $a \neq 0$. The idea is to solve quadratic equations by factoring.

To solve quadratic equations by factoring:
1. Set the equation equal to zero.
2. Factor.
3. Set each factor equal to zero and solve.

Examples

10. Solve: $x^2+4x=5$

Steps	Reasons
$x^2+4x=5$	Since there is an exponent for the variable:
$x^2+4x-5=0$	1. Set the equation equal to zero.
$(x+5)(x-1)=0$	2. Factor.
$x+5=0 \quad x-1=0$	3. Set each factor equal to zero and solve.
$x=-5 \ or \ x=1$	
$x=-5,1$	There are two solutions to the equation.

11. Solve: $x^3 - 5x^2 + 6x = 0$

Steps	Reasons
$x^3 - 5x^2 + 6x = 0$	The equation is already set equal to zero. So, just factor the left.
$x(x^2 - 5x + 6) = 0$	x is a common factor.
$x(x-2)(x-3) = 0$	The factors of 6 that add up to –5 are –2 and –3.
$x = 0 \;\; or \;\; x - 2 = 0 \;\; or \;\; x - 3 = 0$ $x = 2 \qquad x = 3$	Set each factor equal to zero.
$x = 0,2,3$	These are the solutions.

12. Solve: $4x^2 - x = 5$

Steps	Reasons
$4x^2 - x = 5$	Since there is an exponent for the variable: 1. Set the equation equal to zero by subtracting 5 on both sides.
$4x^2 - x - 5 = 0$	2. Factor using $ax^2 + bx + c$ check the following possibilities. $(4x-5)(x+1)$ $(2x-5)(2x+1)$
$(4x-5)(x+1) = 0$	$(4x+5)(x-1)$ $(2x+5)(2x-1)$ $(4x-1)(x+5)$ $(2x-1)(2x+5)$
$4x - 5 = 0 \;\; or \;\; x + 1 = 0$ $4x = 5 \qquad x = -1$ $x = \frac{5}{4}$	$(4x+1)(x-5)$ $(2x-1)(2x+5)$ 3. Set each factor equal to zero and solve.
$x = -1, \frac{5}{4}$	There are two solutions to the equation.

13. Solve: $(2x+5)(x+1)=-1$

Steps	Reasons
$(2x+5)(x+1)=-1$	Since the equation is not yet set equal to zero, we should multiply on the left using FOIL.
$2x^2+2x+5x+5=-1$	Collect like terms and set the equation equal to zero.
$2x^2+7x+6=0$	Factor using the methods for ax^2+bx+c $(x+1)(2x+6)$ $(x+2)(2x+3)$ $(x+6)(2x+1)$ $(x+3)(2x+2)$ $(x-1)(2x-6)$ $(x-2)(2x-3)$ $(x-6)(2x-1)$ $(x-3)(2x-2)$
$(x+2)(2x+3)=0$	
$x+2=0$ $2x+3=0$	Check the possibilities to find the factors. Set each factor equal to zero and solve.
$x=-2$ $2x=-3$	
$x=-2$ *or* $x=-\frac{3}{2}$	
$x=-2,-\frac{3}{2}$	There are two solutions to the equation.

Since we can solve equations that have exponents for the variables, we can do word problems that result in this new type of equation. Below are a couple of examples of translation problems that result in degree two equations.

14. The sum of two numbers is six. The sum of the squares of the two numbers is twenty. Find the two numbers.

Steps	Reasons
Sum of two numbers is 6	List information and identify the question.
Sum of squares of two numbers is 20	
Find the numbers??? x = one number 6 – x = the other number	Here we have two unknowns. • Let x = one of them is easy. • Because the sum is 6 the other number is 6 – x Sum – first number (Remember: This is the relation for two numbers adding up to a given sum)

Steps	Reasons
Sum of squares of two numbers is 20 One number² + other number² = 20 x² + (6 – x)² = 20	Use the other information to write an equation.
$x^2 + (6-x)^2 = 20$	Solve the equation.
$x^2 + 36 - 12x + x^2 = 20$	You can multiply (6 – x)(6 – x) on the side.
$2x^2 - 12x + 36 = 20$	$(6-x)(6-x)$ $36 - 6x - 6x + x^2$ Use FOIL $36 - 12x + x^2$
$2x^2 - 12x + 16 = 0$	
$2(x^2 - 6x + 8) = 0$	Set the equation equal to zero and factor. Since $x^2 - 6x + 8$ does not have a number in front of x^2, we find the factors of 8 whose sum is -6. Use -2 and -4 to factor.
$2(x-2)(x-4) = 0$	
$x - 2 = 0 \quad x - 4 = 0$	
$x = 2, \quad x = 4$	
If x = 2, then 6 – x = 6 – 2 = 4	Find the other number using 6 – x.
The numbers are 2 and 4.	State your answer.

15. The sum of two numbers is thirteen. The product of the two numbers is forty. Find the two numbers.

Steps	Reasons
Sum of two numbers is 13 The product of the two numbers is forty. Find the numbers???	List information and identify the question.
x = one number 13 – x = the other number	Here we have two unknowns. • Let x = one of them is easy. • Because the sum is 13 the other number is 13 – x Sum – first number (Remember: This is the relation for two numbers adding up to a given sum)

The product of the two numbers is forty.
One number $\times$ other number = 40
x times $13-x$ =40

Use the other information to write an equation.

$$x(13-x)=40$$

Solve the equation.

$$13x-x^2=40$$

Multiply.

$$-x^2+13x-40=40-40$$

$$-x^2+13x-40=0$$

Set equal to zero.

$$-\left(x^2-13x+40\right)=0$$

I like to factor out the negative sign so that the first term is positive.

$$-(x-8)(x-5)=0$$

Set each factor equal to zero.

$$x-8=0$$
$$x=8$$
$$x-5=0$$
$$x=5$$

If x = 8
then 13 – x = 13 – 8 = 5

Find the other number using 13 – x.

The numbers are 8 and 5.

State your answer.

Exercises

Factor:

1. $12x^3 + 6x^2 - 18x$
2. $50x^3 + 30x^2 - 70x$
3. $20x^5 - 16x^4 - 12x^3$
4. $21x^5 - 14x^4 - 7x^3$
5. $33x^6 + 55x^4 - 44x^2$
6. $56x^6 - 16x^4 - 32x^2$
7. $63x^5 - 18x^4 + 54x^2$
8. $25y^7 - 15y^6 - 45y^3$
9. $40y^6 - 49y^4 - 35y^2$
10. $11y^7 - 99y^6 - 66y^5$
11. $x^2 + 5x + 6$
12. $x^2 + 4x + 3$
13. $x^2 + 9x + 8$
14. $x^2 + 6x + 8$
15. $x^2 - 2x - 8$
16. $x^2 - 2x + 12$
17. $x^2 + 2x - 7$
18. $x^2 - 3x - 18$

19. $x^2 - 7x + 12$

20. $x^2 - 8x + 15$

21. $x^2 - x - 20$

22. $x^2 - 2x - 3$

23. $x^2 + 4x - 21$

24. $x^2 + 3x - 10$

25. $x^2 - 5x + 4$

26. $x^2 - 12x + 35$

27. $x^2 + 11x + 18$

28. $x^2 + 14x + 40$

29. $x^2 - 5x - 24$

30. $x^2 - x - 42$

31. $x^2 + x - 30$

32. $x^2 + 5x + 2$

33. $x^2 - x - 9$

34. $x^2 + 4x - 12$

35. $x^2 + 16x + 64$

36. $x^2 + 12x + 36$

37. $2x^2 + 5x - 3$

38. $3x^2 + 6x + 2$

39. $3x^2 - 16x + 5$

40. $2x^2 + 5x + 2$

41. $3x^2 - 11x - 10$

42. $3x^2 - 4x + 1$

43. $2x^2 + x - 3$

44. $5x^2 + 9x - 2$

45. $5x^2 - 13x - 6$

46. $2x^2 - 7x + 3$

47. $6x^2 + x - 1$

48. $6x^2 + 5x - 6$

49. $4x^2 - 5x - 6$

50. $4x^2 - 8x + 3$

51. $4x^2 - 4x - 15$

52. $6x^2 - 17x - 3$

Factor Completely:

53. $5x^3 + 20x^2 + 15x$

54. $10x^4 - 10x^3 - 60x^2$

55. $3x^4 - 9x^3 - 30x^2$

56. $7x^5 - 21x^4 + 14x^3$

57. $4x^5 - 24x^4 + 32x^3$

58. $5x^5 + 10x^4 - 40x^3$

59. $20x^5 + 100x^4 + 80x^3$

60. $2x^6 - 16x^5 + 30x^4$

Solve:

61. $(x - 9)(x - 7) = 0$

62. $(x + 6)(x - 5) = 0$

63. $(2x + 3)(3x - 1) = 0$

64. $(5x - 9)(2x + 7) = 0$

65. $x^2 - 5x + 4 = 0$

66. $x^2 + 6x + 5 = 0$

67. $x^2 + 4x - 21 = 0$

68. $x^2 - 3x - 40 = 0$

69. $x^2 = 5x + 6$

70. $x^2 = 6x - 8$

71. $x^2 - x = 12$

72. $x^2 = 2x + 8$

73. $x^2 - 9x = -18$

74. $x^2 - 4x = 21$

75. $x^3 - 3x^2 + 2x = 0$

76. $x^3 + 5x^2 + 6x = 0$

77. $x^3 - x^2 - 20x = 0$

78. $x^3 - 11x^2 + 10x = 0$

79. $3x^2 - 5x - 2 = 0$

80. $2x^2 - 7x + 3 = 0$

81. $2x^2 + x - 3 = 0$

82. $3x^2 + x - 2 = 0$

Solve by using an appropriate equation:

83. The sum of two numbers is eleven. The product of the two numbers is thirty. Find the two numbers.

84. The sum of two numbers is fifteen. The product of the two numbers is thirty-six. Find the two numbers.

85. The sum of two numbers is eight. The sum of the squares of the two numbers is thirty-four. Find the two numbers.

86. The sum of two numbers is seven. The sum of the squares of the two numbers is twenty-nine. Find the two numbers.

Quadratic Formula

Given a polynomial of the form $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is called the quadratic formula. It is derived by solving for $ax^2 + bx + c = 0$ by "completing the square," which is beyond the scope of this class.

<u>Examples:</u>

1. Solve $3x^2 - 2x - 5 = 0$ using the quadratic formula.

Steps	Reasons
$3x^2 - 2x - 5 = 0$ $a = 3,\ b = -2,\ c = -5$	Identify the a,b, and c from the general form: $ax^2 + bx + c = 0$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the quadratic formula.
$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-5)}}{2(3)}$	Replace a,b, and c with the numbers.
$x = \frac{2 \pm \sqrt{4 + 60}}{6}$	Simplify.
$x = \frac{2 \pm \sqrt{64}}{6}$	
$x = \frac{2 \pm 8}{6}$	
$x = \frac{2 + 8}{6} = \frac{10}{6} = \frac{5}{3}$ $x = \frac{2 - 8}{6} = \frac{-6}{6} = -1$	Write out the + and the $-$.
$x = -1, \frac{5}{3}$	Write the answers. The same answer can be found by factoring to solve.

2. Solve $3x^2 = x + 5$ using the quadratic formula.

Steps	Reasons
$3x^2 = x + 5$	Get the polynomial on one side and zero on the other in order to identify the a,b, and c by subtracting x and 5 from both sides. It is usually easier to keep the x^2 term positive.
$3x^2 - x - 5 = 0$	
$a = 3, \quad b = -1, \quad c = -5$	
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the quadratic formula.
$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)}$	Replace a,b, and c with the numbers.
$x = \dfrac{1 \pm \sqrt{1+60}}{6}$	Simplify by performing the arithmetic. Careful of the signs!
$x = \dfrac{1 \pm \sqrt{61}}{6}$	The square root cannot be evaluated or simplified. So, we stop.
$x = \dfrac{1 \pm \sqrt{61}}{6} \quad or \quad \left\{ \dfrac{1-\sqrt{61}}{6}, \dfrac{1+\sqrt{61}}{6} \right\}$	You may write the answer either way.

The above example shows that we can use the quadratic formula to solve for equations that we cannot solve by factoring. If we are asked to solve a quadratic equation ($ax^2 + bx + c = 0$) and the problem is hard to factor, then we need to try the quadratic formula quickly. It may be that the equation cannot be solved by factoring.

3. Solve $2x^2 - 4x = 7$ using the quadratic formula.

Steps	Reasons
$2x^2 - 4x = 7$	
$2x^2 - 4x - 7 = 0$	Get the polynomial on one side and zero on the other in order to identify the a,b, and c.
$a = 2,\ \ b = -4,\ \ c = -7$	
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the quadratic formula.
$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-7)}}{2(2)}$	Replace a,b, and c with the numbers.
$x = \dfrac{4 \pm \sqrt{16 + 56}}{4}$	Simplify.
$x = \dfrac{4 \pm \sqrt{72}}{4}$	
$x = \dfrac{4 \pm \sqrt{36}\sqrt{2}}{4}$	Simplify the square root by finding a factor that is a perfect square.
$x = \dfrac{4 \pm 6\sqrt{2}}{4}$	
$x = \dfrac{2\left(2 \pm 3\sqrt{2}\right)}{4}$	Simplify the fraction by factoring in the numerator and canceling the common factor. We must factor to cancel because of the + or – signs.
$x = \dfrac{2 \pm 3\sqrt{2}}{2}$	
$x = \dfrac{2 \pm 3\sqrt{2}}{2} \quad or \quad \left\{ \dfrac{2 - 3\sqrt{2}}{2}, \dfrac{2 + 3\sqrt{2}}{2} \right\}$	You may write the answer either way.

Exercises

Solve by using the quadratic formula:

1. $3x^2 - 2x - 5 = 0$
2. $2x^2 + 3x - 2 = 0$
3. $5x^2 + 3x - 1 = 0$
4. $4x^2 + 5x = 2$
5. $3x^2 - 7x = -3$
6. $2x^2 - 9x + 5 = 0$
7. $x^2 = 3x - 1$
8. $x^2 = -5x - 1$
9. $3x^2 - x - 7 = 0$
10. $2x^2 + 6x - 9 = 0$
11. $3x^2 - 2x = 9$
12. $4x^2 + 6x - 9 = 0$
13. $12x^2 = 4x + 1$
14. $9x^2 - 3x - 5 = 0$
15. $4x^2 + 2x - 1 = 0$
16. $2x^2 - 3x - 5 = 0$
17. $3x^2 - 4x - 4 = 0$
18. $2x^2 + 10x + 1 = 0$

19. $4x^2 - 14x + 3 = 0$

20. $x^2 - 6x - 7 = 0$

21. $x^2 - 2x - 5 = 0$

5

We formalize the relationship between two variables by defining a function. We explore linear, quadratic, exponential, and logarithmic functions. Students learn to graph these four functions and analyze associated application problems. Systems of linear functions and their applications are studied.

Course Outcome:

- Demonstrate mastery of algebraic skills
- Demonstrate understanding of the concepts of functions and related applications
- Recognize and apply mathematical concepts to real-world situation

5.1 Graphs and Functions

The rectangular coordinate system and ordered pairs are defined. Function notation and the vertical line test for functions are introduced. Linear and parabolic functions are graphed by picking points. The discussion includes applied problems for functions.

5.2 Graphing Linear Equations

Linear equations and three different methods for graphing are explained: picking points, x-intercept/y-intercept, and slope/y-intercept. The slope of a line is described geometrically and algebraically. Vertical lines and horizontal lines are discussed in terms of equation, slope, and graph.

5.3 Solving Systems of Linear Equations

Three methods for solving systems of equations are discussed: graphing, substitution, and addition. Solving by graphing is used to discuss the types of solutions. The focus is on solving using the substitution and addition methods. Applications of solving systems of equations are explored.

5.4 Quadratic Functions

Degree two or quadratic functions are developed. Students will be able to find the vertex, x-intercepts, y-intercept, and graph quadratic functions. Application problems include finding the vertex to determine the minimum or maximum of the quadratic function.

5.5 Exponential Functions

Exponential functions are defined. Students learn to graph exponential functions and solve some applied problems involving exponential functions.

5.6 Logarithmic Functions

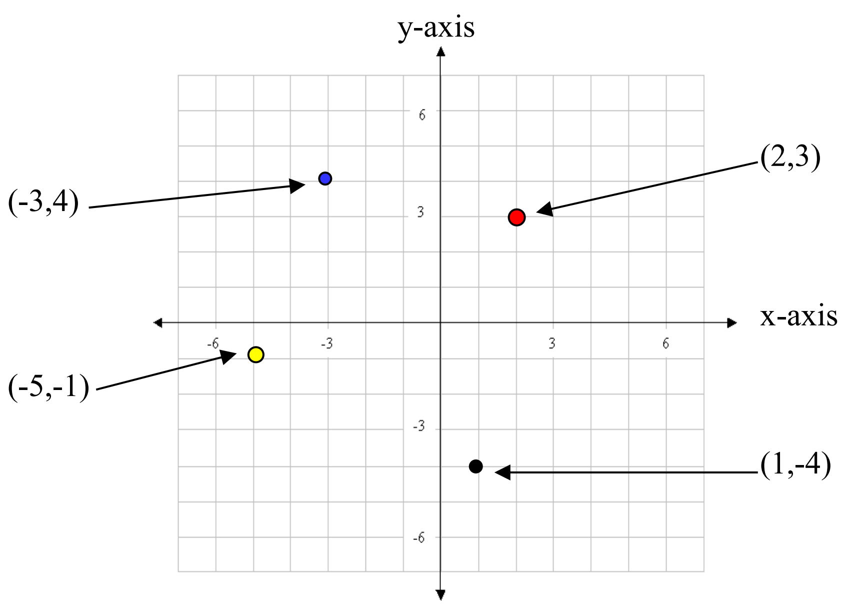
Logarithmic functions are defined. Students should be able to graph logarithmic functions and solve some applied problems involving logarithmic functions.

The idea behind graphing is that we are relating two variables. The rectangular coordinate system is made up of two real number lines that are perpendicular.

Ordered pairs have two coordinates and are used to represent the different points on the coordinate plane:
1. The first coordinate (or abscissa) refers to the horizontal axis (or x-axis).
2. The second coordinate (or ordinate) refers to the vertical axis (or y-axis).

Graphs of Order Pairs

(2,3) 2 to the right on the horizontal axis and 3 up on the vertical axis
(-3,4) 3 to the left on the horizontal axis and 4 up on the vertical axis
(-5,-1) 5 to the left on the horizontal axis and 1 down on the vertical axis
(1,-4) 1 to the right on the horizontal axis and 4 down on the vertical axis



The origin is the point in the middle where the two axes cross with coordinates (0,0).

Function Notation allows us to describe a certain type of relation.
The idea is to define a function f, g, or h in terms of a variable and arithmetic operations. Here we are evaluating functions for a specific value by replacing the variable x with the number and evaluating the expression.

Example

1. Given f(x) = x^2 +1, find f(1), f(3), and f(-2).

Steps	Reasons
f(1)	
$f(x) = x^2 + 1$	Write the function.
$f(1) = 1^2 + 1$	Replace x with 1.
$f(1) = 1 + 1$	
	Simplify.
$f(1) = 2$	
f(3)	
$f(x) = x^2 + 1$	Write the function.
	Replace x with 3.
$f(3) = 3^2 + 1$	
$f(3) = 9 + 1$	
	Simplify.
$f(3) = 10$	
f(-2)	
$f(x) = x^2 + 1$	Write the function.
$f(-2) = (-2)^2 + 1$	Replace x with -2. Use parentheses to replace the number with the variable.
$f(-2) = 4 + 1$	
	Simplify.
$f(-2) = 5$	

There really is a great deal that can be said about functions. Functions can be graphed by writing points as $(x, f(x))$ much as we graph with ordered pairs (x, y). To graph we can pick some x-values and then determine their corresponding f(x) values.

Examples

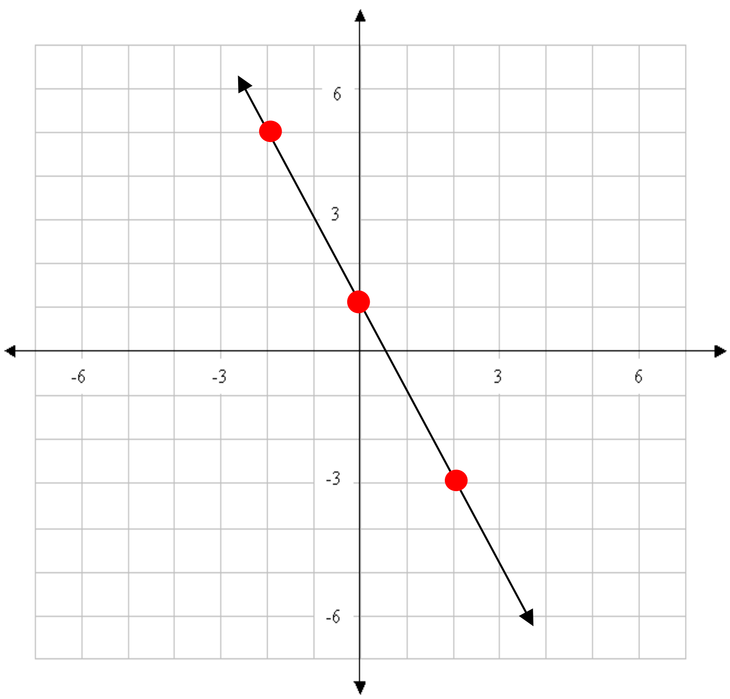
2. Graph $f(x)=-2x+1$ using x = -2,0,2

x	$f(x)$
-2	5
0	1
2	-3

$$\frac{f(x)=-2x+1}{f(-2)=-2(-2)+1=4+1=5}$$
$$f(0)=-2(0)+1=1$$
$$f(2)=-2(2)+1=-4+1=-3$$

Here we have what looks like a linear equation, which we can graph by picking a few points and making a table. Here -2, 0, and 2 were chosen.

Graph the points $(x, f(x))$ from the above table and draw the line that connects them



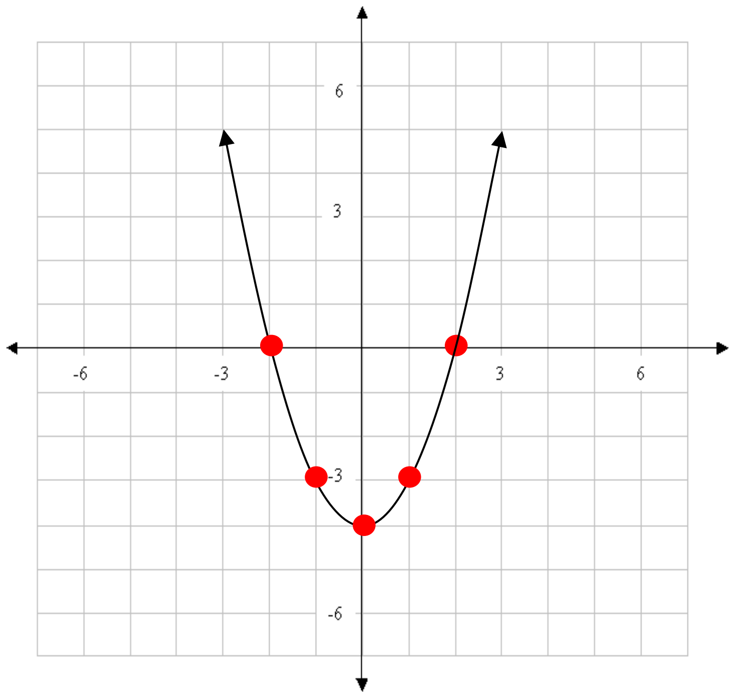
3. Graph $f(x)=x^2-4$ using x = -2,-1,0,1,2

x	$f(x)$
-2	0
-1	-3
0	-4
1	-3
2	0

$$f(x)=x^2-4$$
$$f(-2)=(-2)^2-4=4-4=0$$
$$f(-1)=(-1)^2-4=1-4=-3$$
$$f(0)=(0)^2-4=0-4=-4$$
$$f(1)=(1)^2-4=1-4=-3$$
$$f(2)=(2)^2-4=4-4=0$$

We may not know the shape of the graph. Just trying some x-values is a good strategy until we know more about the shape of this type of function. Here we are just trying some x-values around zero: -2, -1, 0, 1, and 2, which turns out to be a very good choice.

Graph the points $\left(x, f(x)\right)$ from the above table.



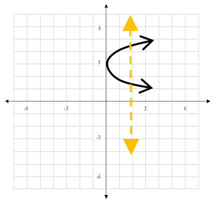
Here the shape is $\cup$, which is called a parabola. It turns out that all quadratic functions, which have the form $f(x) = ax^2 + bx + c$, have graphs that are parabolas. The x^2 is the most obvious difference with the linear functions.

Aside from having its own notation, there is a stricter definition of functions that states for every x value there is at most one y value. So, if a graph has two points such as (2,3) and (2,5), it is not the graph of a function because the x=2 is related to two different y-values. From this restriction, we get the following:

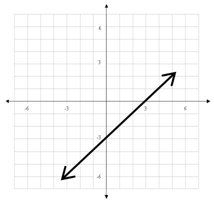
Vertical Line Test:
If no vertical line intersects the graph more than once, then it is the graph of a function.

Examples

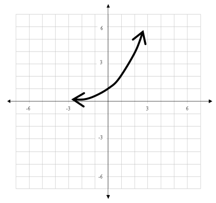
4. Which of the following are graphs of functions?



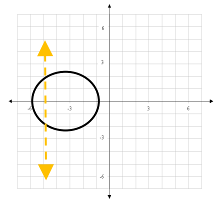
Not a function - we can draw a vertical line that intersects the graph more than once.



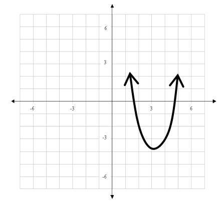
Function - we cannot draw a vertical line that intersects the graph more than once.



Function - we cannot draw a vertical line that intersects the graph more than once.



Not a function - we can draw a vertical line that intersects the graph more than once.



Function - we cannot draw a vertical line that intersects the graph more than once.

5. The cost of a new car can be estimated by the function:
$C(x) = -2.4x^2 + 950x + 8{,}500$ where *x* is the number of years after 1990.

a. Use the formula to estimate the cost of a new car in 2005.
b. If the actual cost of a new car in 2005 was $21,750, does the formula underestimate or overestimate the actual price? By how much?

Steps	Reasons
$C(x) = -2.4x^2 + 950x + 8{,}500$	2005 – 1990 = 15
$C(15) = -2.4(15)^2 + 950(15) + 8{,}500$	2005 is 15 years after 1990. So, use x=15.
$C(15) = 22{,}210$	Use a calculator to evaluate the function.
a. In 2005 the new car will cost about $22,100.	State the answer.
b. The function overestimates the cost of the new car by $460.	Since $22,100 is more than the actual price of $21,750, the formula overestimates the actual price by 22,100 – 21,750 = 460.

Exercises

Using a straightedge, please draw the x-axis and y-axis on a sheet of graphing paper. Then, plot each of the following.

1. $(2,5)$
2. $(-4,3)$
3. $(-3,-5)$
4. $(5,0)$
5. $(3,-1)$
6. $(-3,2)$
7. $(0,0)$
8. $(-4,-2)$
9. $(2.5,3.5)$
10. $(1.5,-2.5)$
11. $\left(\frac{1}{2},\frac{3}{4}\right)$
12. $\left(-\frac{1}{3},\frac{2}{3}\right)$

Evaluate the function for the following values:

13. $f(x) = 3x - 5$
    a. $f(2)$
    b. $f(10)$
    c. $f(-3)$
    d. $f(0)$

14. $h(x) = 8x - 5$
    a. $h(4)$
    b. $h(-5)$
    c. $h\left(\frac{1}{4}\right)$
    d. $h(0)$

15. $g(x) = x^2 + 5x$
    a. $g(2)$
    b. $g(-3)$
    c. $g(0)$
    d. $g\left(\frac{1}{5}\right)$

16. $f(x) = 2x^2 - 3x$
    a. $f(4)$
    b. $f(-5)$
    c. $f(0)$
    d. $f(-3)$

17. $f(x) = 3x^2 + 8$
    a. $f(3)$
    b. $f(-2)$
    c. $f(0)$
    d. $f(-5)$

18. $g(x) = 5x^2 - 3$
    a. $g(1)$
    b. $g(-2)$
    c. $g(0)$
    d. $g\left(\frac{1}{2}\right)$

19. $h(x) = \sqrt{x + 9}$
    a. $h(7)$
    b. $h(-5)$
    c. $h(0)$
    d. $h(91)$

20. $f(x) = \sqrt{x + 4}$
    a. $f(21)$
    b. $f(-3)$
    c. $f(0)$
    d. $f(285)$

Graph the following functions by first finding the value of the function for the x-values of -2,-1,0,1,2 and then graphing those points. In the end the points should be connected to show the appropriate shape.

21. $f(x) = 3x - 1$

22. $f(x) = 2x + 1$

23. $f(x) = -2x + 1$

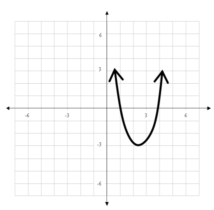
24. $h(x) = \frac{1}{2}x + 3$

25. $g(x) = x^2 + 1$

26. $g(x) = x^2 - 2$

Which of the following are graphs of functions?

27.



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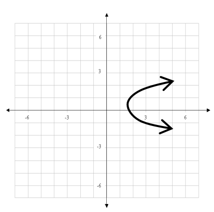

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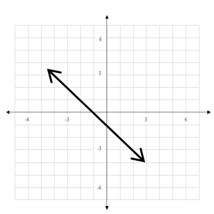

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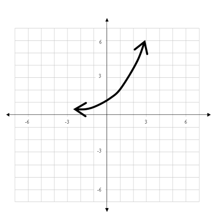

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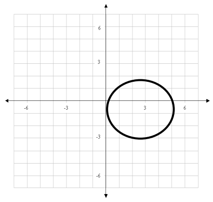

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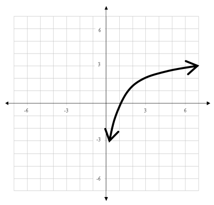

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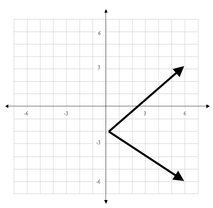

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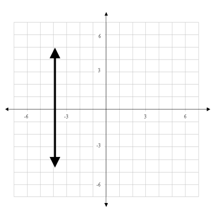

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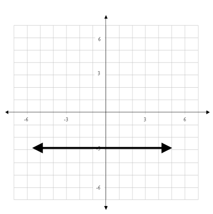


Answer the following:

By Will Tenney

36. The weight of water in a container is related to the volume of water in the container. A rectangular swimming pool with length 15 meters, width 5 meters, and a constant depth is being filled with fresh water. By constant depth we mean that there is no shallow end or deep end. Instead the height or depth of the water is the same throughout the pool. The weight of the water in kilograms in the pool is $\mathrm{W(x)} = 750\mathrm{x}$ where x is the depth of water in the pool in centimeters.
    a. What is the weight of the water when the depth of the water in the pool is 10 cm?
    b. What is the weight of the water in the swimming pool when the depth of the water is 250 cm?

37. The weight of water in a container is related to the volume of water in the container. A rectangular swimming pool with length 15 meters, width 5 meters, and a constant depth is being filled with salt water. By constant depth we mean that there is no shallow end or deep end. Instead the height or depth of the water is the same throughout the pool. The weight of the water in kilograms in the pool is $\mathrm{W(x)} = 772.5\mathrm{x}$ where x is the depth of water in the pool in centimeters.
    a. What is the weight of the salt water when the depth of the water is 10 cm?
    b. What is the weight of the salt water in the swimming pool when the depth of the water is 250 cm?

38. The cost of a new computer can be estimated by the function: $C(x) = -1.8x^2 + 80x + 750$ where *x* is the number of years after 1995.
    a. Use the formula to estimate the cost of a new computer in 2009.
    b. If the actual cost of a new computer in 2009 was $1450, does the function underestimate or overestimate the actual price? By how much?

39. The cost of sending a child to an elite private college can be estimated by the function: $C(x) = 18x^2 + 700x + 2000$ where *x* is the number of years after 1980.
    a. Use the formula to estimate the cost of sending a child to an elite private college in 2015.
    b. If the actual cost of sending a child to an elite private college in 2015 was $52,000, does the function underestimate or overestimate the actual price? By how much?

Solutions to equations with two variables are ordered pairs. Since there can be an infinite number of solutions it may be impossible to list all the solutions. Instead we may show the solutions by drawing a graph of the solutions on the rectangular coordinate system. To check to see if an individual point is a solution, replace x and y in the equation with the appropriate number from the ordered pair (x,y).

(3,2) is a solution to 5x – 4y = 7
Check:

$$5(3)-4(2)=7$$
$$15-8=7$$
$$7=7 \ \ True$$

(1,-5) is not a solution to y = 2x – 6.
Check:

$$-5=2(1)-6$$
$$-5=2-6$$
$$-5=-4 \ \ False$$

Examples

1. Is (5,3) a solution to y = 2x – 7?

Steps	Reasons
5 is x and 3 is y	Check by evaluating the equation for these values of x and y.
y = 2x – 7 3 = 2(5) – 7 3 = 10 –7 3 = 3	If the replacement is true, then the ordered pair is a solution. If the replacement is false, then the ordered pair is not solution.
Yes, (5,3) is a solution to y = 2x –7	Say yes or no.

When we write equations with two variables, there can be an infinite number of different combinations that are solutions depending on the choice of x. Since all of the solutions cannot be listed, we graph pictures of the solutions called graphs.

Linear equations in two variables are equations whose graphs are lines. Every point on the line will be a solution to the equation.

There are three main ways to graph linear equations. In the beginning we just pick a few points. Two points determine a line and another point will help to check.

Graphing by picking points

1. Pick three x-values. 2,0,-2 or 3,0,-3 are often good choices. Three points around x=0 that avoid fractions is best.
2. Calculate the y-value.
3. Graph the points on a set of axes.
4. Draw the line. Use a straight-edge for all lines including axes.

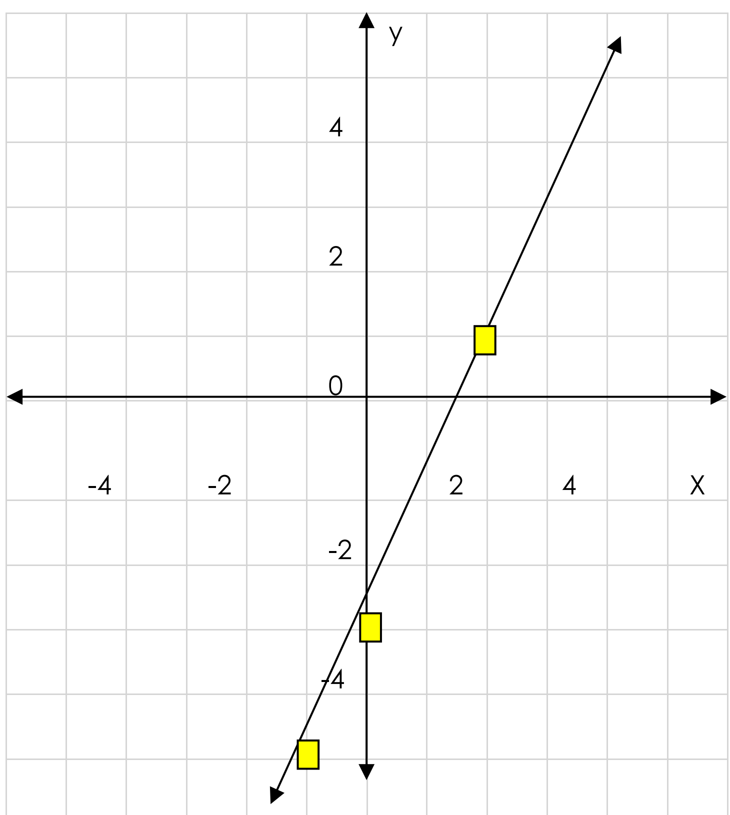
Examples

2. Graph: $y = 2x – 3$

Make a table. You pick the x-values. -2,0,2 were chosen, but other x-values will yield the same line.
Evaluate $y = 2x – 3$ for the x-values that you choose. Note that $y = 2x – 3$. So, evaluating $2x – 3$ for x-values gives y-values and points on the line.

X	Y	Y = 2x – 3
– 2	-7	$2(–2) – 3 = – 4 – 3 =–7$
0	-3	$2(0) – 3 = 0 – 3 = –3$
2	1	$2(2) – 3 = 4 – 3 = 1$
–1	-5	$2(–1) – 3= –2 – 3 = –5$

From the chart the points (–2,–7), (0, –3), (2,1), and (–1,–5) are all points on the line. There are yellow squares over the points from the graph. Draw the line that connects the points.



3. Graph $y=-\frac{1}{3}x+2$

X	Y
– 3	3
0	2
3	1

$y=-\frac{1}{3}x+2$

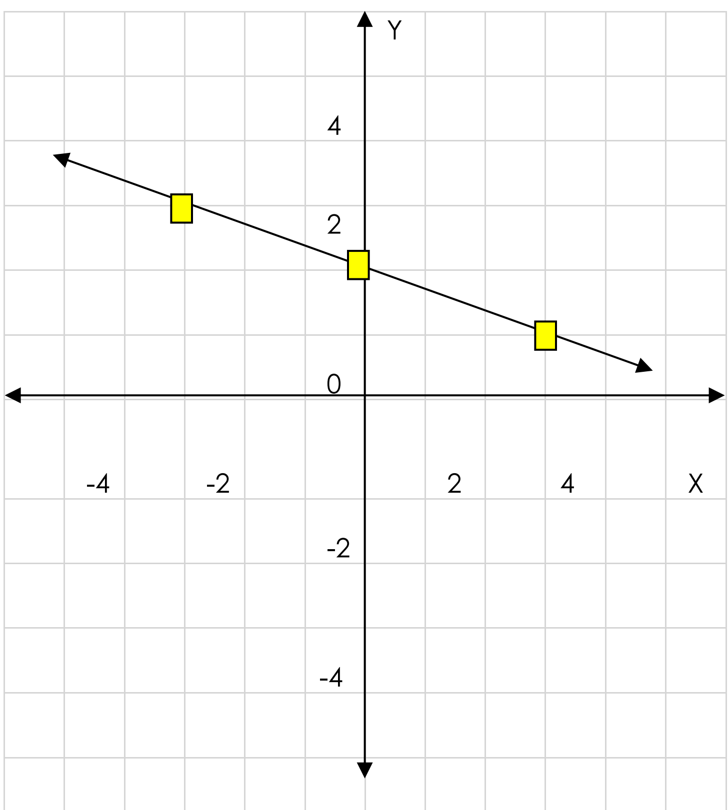
$-\frac{1}{3}(-3)+2=1+2=3$

$-\frac{1}{3}(0)+2=0+2=2$

$-\frac{1}{3}(3)+2=-1+2=1$

Make a table. You pick the x-values. I picked multiples of three (denominator) to avoid fractions. Evaluate $-\frac{1}{3}x+2$ for the x-values that you choose. Note that $y=-\frac{1}{3}x+2$. So evaluating $-\frac{1}{3}x+2$ for x-values gives y-values and points on the line.

The line goes through the points (–3,3), (0,2), and (3,1).

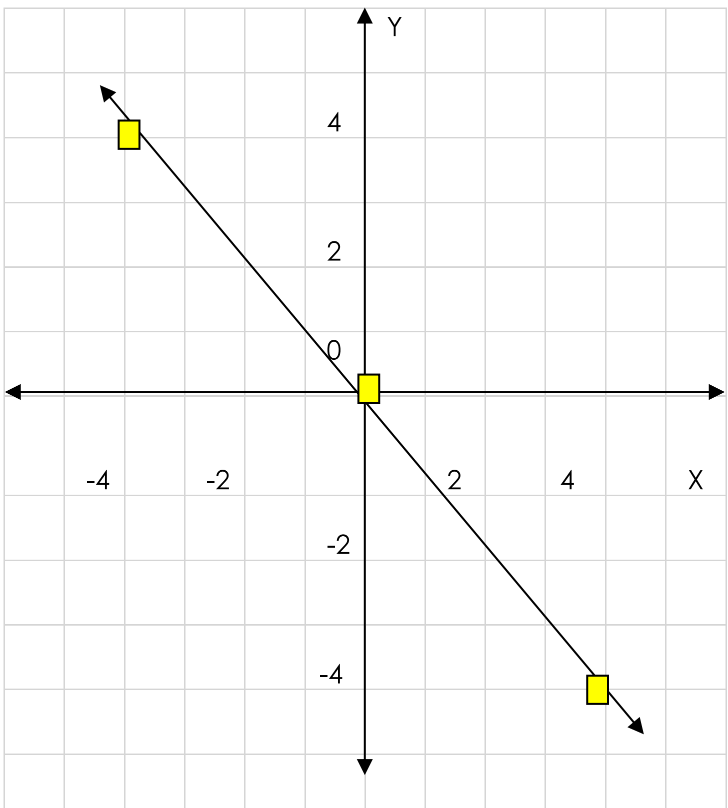


4. Graph: $y = -x$

X	Y	$y = -x$
-4	4	$-(-4) = 4$
0	0	$-0 = 0$
4	-4	-4

Make a table. You pick the x-values. Evaluate $y = -x$ for the x-values that you choose.

$(-4,4)$, $(0,0)$, and $(4,-4)$ are all points on the line.



Because there may be an infinite number of solutions to an equation in two variables, we often draw the solutions. Remember, solutions are ordered pairs. So, the graphs will be on the coordinate plane.

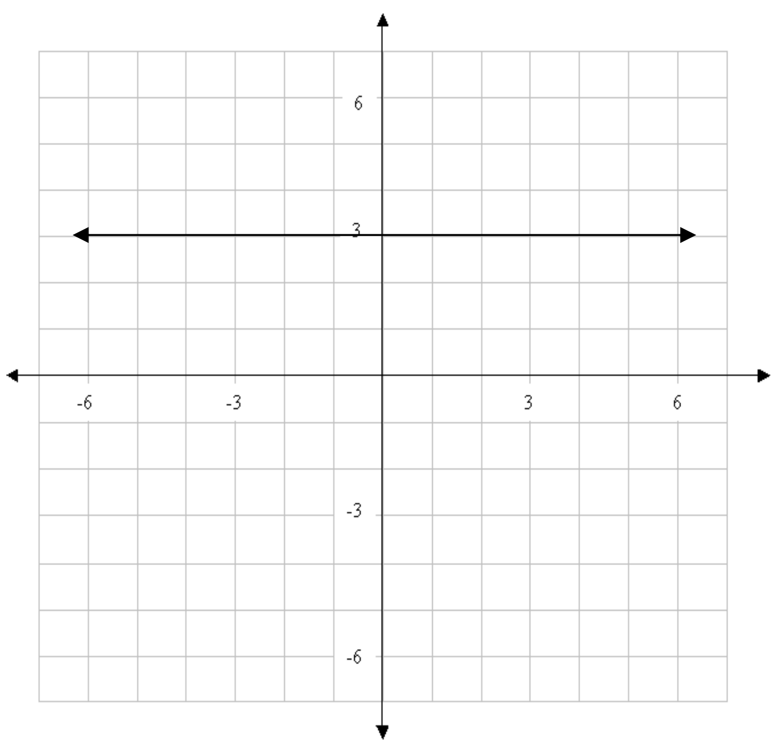
Here we are graphing linear equations, which means our graphs will be lines.

Horizontal lines have the form y = b.

Vertical lines have the form x = a.

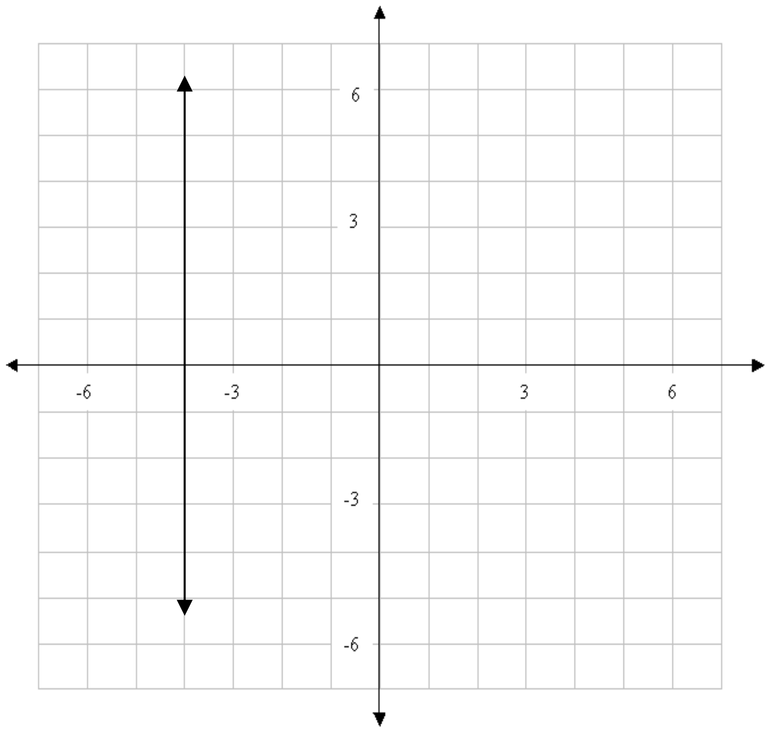
It is best to graph the horizontal and vertical lines by recognizing the form of the equation.

5. Graph y = 3



y = 3 is a horizontal line. Notice that all of the points on the line have a second coordinate of 3. (y=3)

6. Graph x = – 4

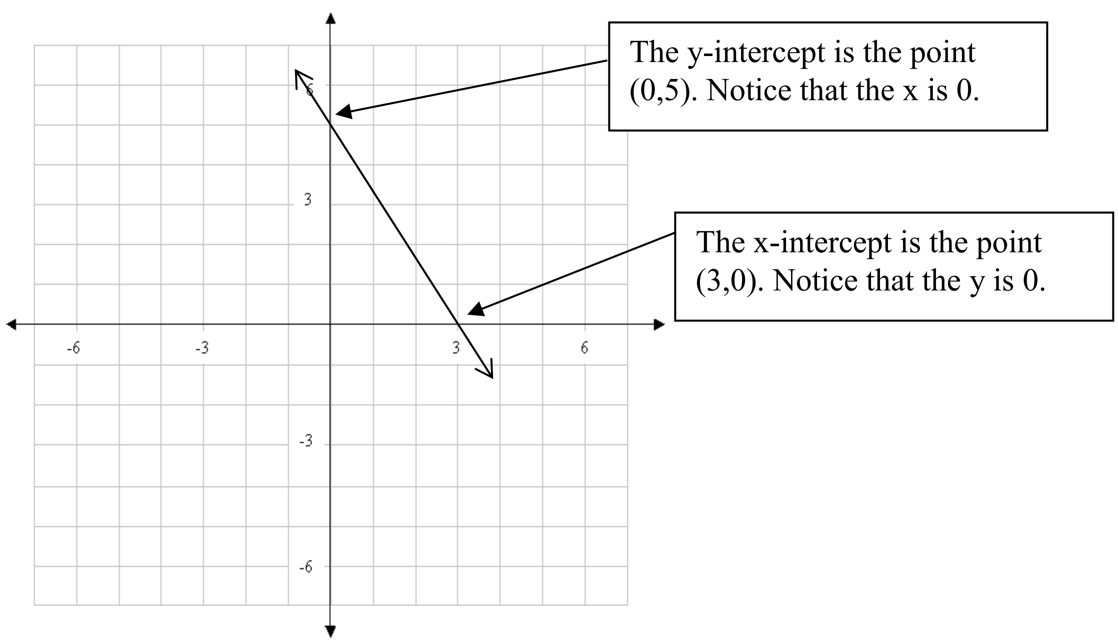


x = – 4 is a vertical line. Notice that all of the points on the line have a first coordinate of – 4. (x = – 4)

The x-intercept is the point where the graph crosses the x-axis. The x-intercept always has second coordinate of 0 (y = 0).

The y-intercept is the point where the graph crosses the y-axis. The y-intercept always has first coordinate of 0 (x = 0).

Look at the x-intercept and y-intercept on the following graph.



To find the x-intercept:
1. Let y = 0
2. Solve for x.
3. Write answer.

To find the y-intercept:
1. Let x = 0
2. Solve for y.
3. Write answer.

Graphing using the x-intercept and y-intercept:
1. Graph the x-intercept and y-intercept.
2. Draw the line through the intercepts.

Vertical and horizontal lines cannot be graphed using this method. Graphing by intercepts works best when the equation has the form $Ax + By = C$.

Examples

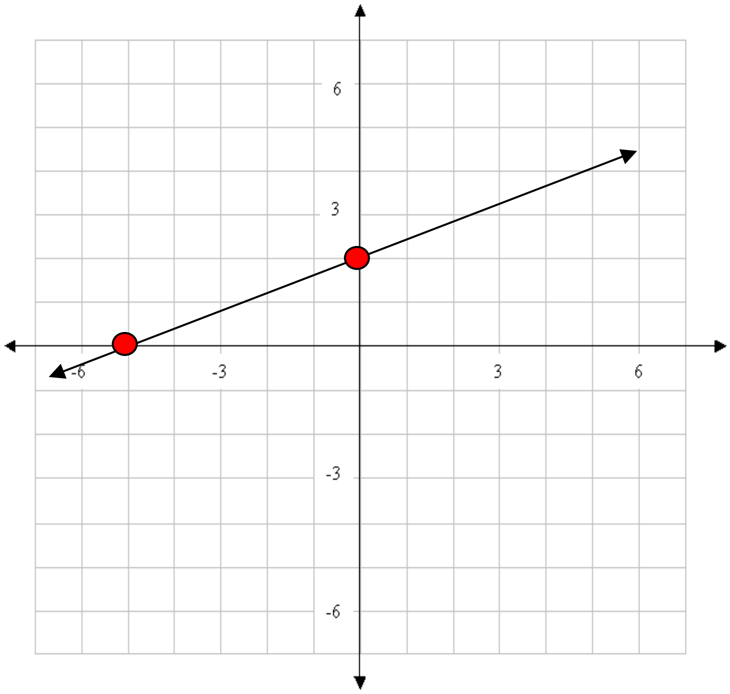
7. Find the x-intercept and y-intercept for 6x –12 y = 24

x-intercept (Let y=0)	y-intercept (Let x=0)
$6x - 12(0) = 24$	$6(0) - 12y = 24$
$6x = 24$	$-12y = 24$
$x = 4 \ \ or \ \ (4,0)$	$y = -2 \ \ or \ \ (0,-2)$

8. Find the x- and y-intercepts and graph 2x – 5y = –10

x-intercept (Let y=0)	y-intercept (Let x=0)
$2x - 5(0) = -10$	$2(0) - 5y = -10$
$2x = -10$	$-5y = -10$
$x = -5 \ \ or \ \ (-5,0)$	$y = 2 \ or \ (0,2)$

Graph the intercepts and then graph the line containing two intercepts. Here we have a second way to graph.



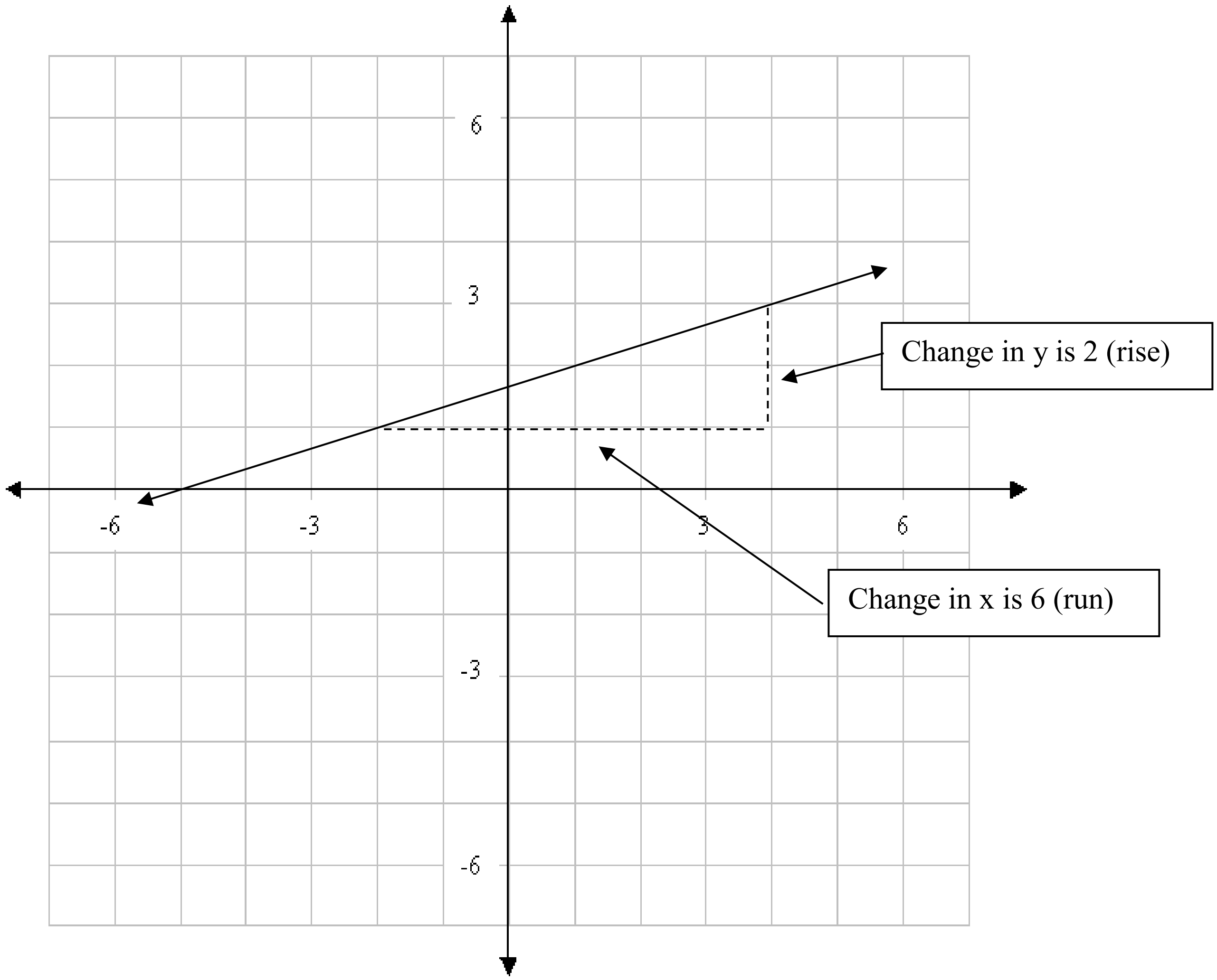
The slope of a line (m) can be thought of as:

1. $m = \dfrac{the\ change\ in\ y}{the\ change\ in\ x}$

2. $m = \dfrac{rise}{run}$

3. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ for any two points on the line (x_1, y_1) and (x_2, y_2)

<u>Look at any line:</u>



The slope of this line is

$$m = \frac{the\ change\ in\ y}{the\ change\ in\ x} = \frac{2}{6} = \frac{1}{3}$$

Examples:

9. Find the slope of the line containing the points (–2,1) and (4,3).
(x_1, y_1) and (x_2, y_2)

Substitute the x and y values into the formula and simplify.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Note: That the two points were on the line above and we got the same slope. Either by looking at the graph or by looking at two points on the line we get the same slope.

10. Find the slope of the line containing the points (5,1) and (5,– 4).

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-4-1}{5-(5)} = \dfrac{-5}{0}$ The slope is undefined.

If you draw the two points, you will see that the points lie on a vertical line.

<u>Vertical lines</u> have undefined slope.

<u>Horizontal lines</u> have slope of zero.

Two lines are parallel lines if they have the same slope (and different y-intercepts.)

<u>A line is in slope-intercept form if it is written as</u>

$y = mx + b$ $\quad$ $m = \text{slope}$
$\quad\quad\quad\quad\quad\;$ $b = \text{y-intercept}$

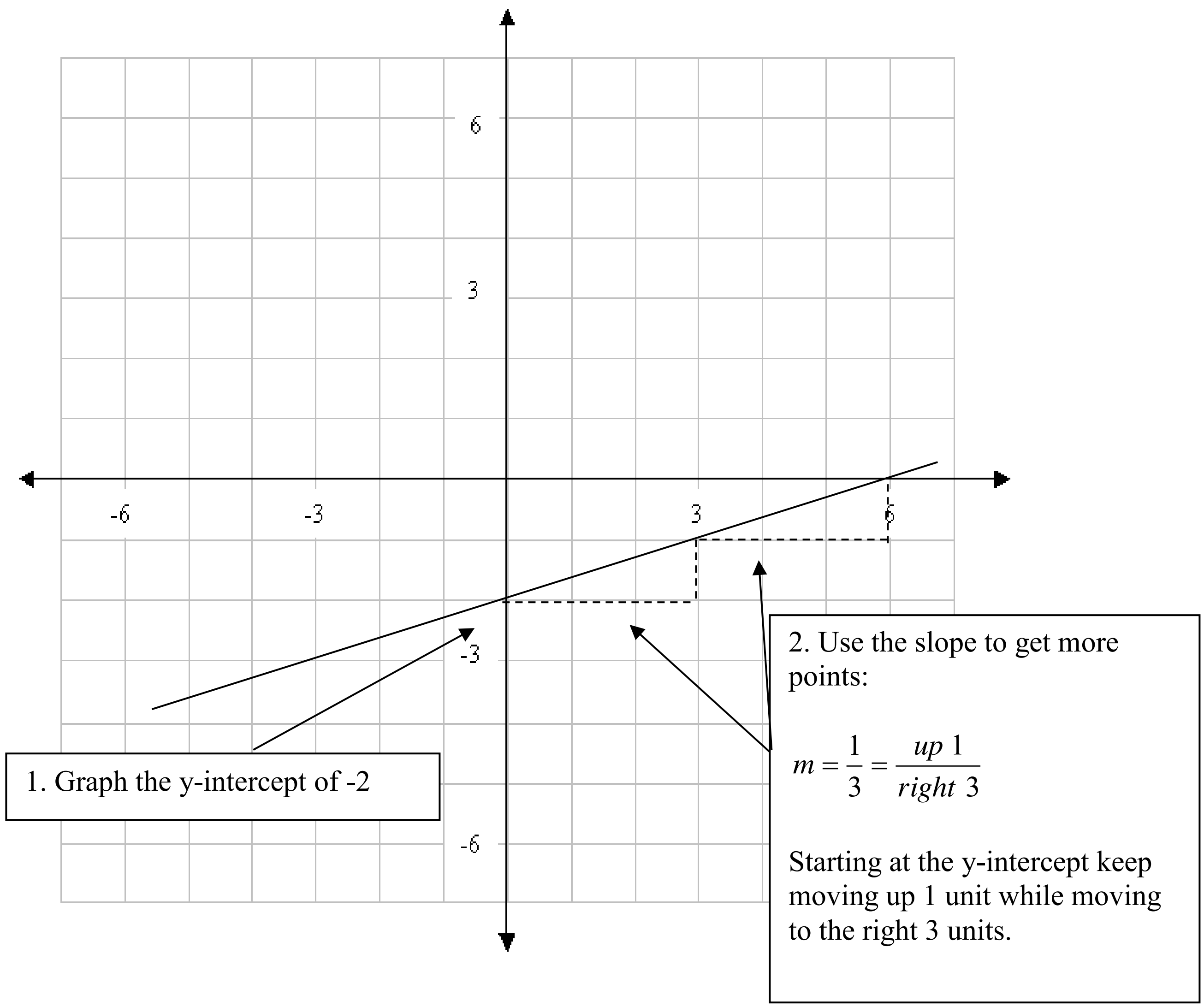
We can read off the information of the slope and y-intercept is in this form. Then we can use that information to graph. Here we have a third way to graph lines.

<u>Steps for graphing using the slope and y-intercept.</u>

1. Get the equation in $y = mx + b$ form to get the slope and y-intercept.
2. Graph the y-intercept.
3. Write the slope as a fraction and use $m = \dfrac{\textit{the change in } y}{\textit{the change in } x}$ which is the same as $m = \dfrac{rise}{run}$ to get more points.

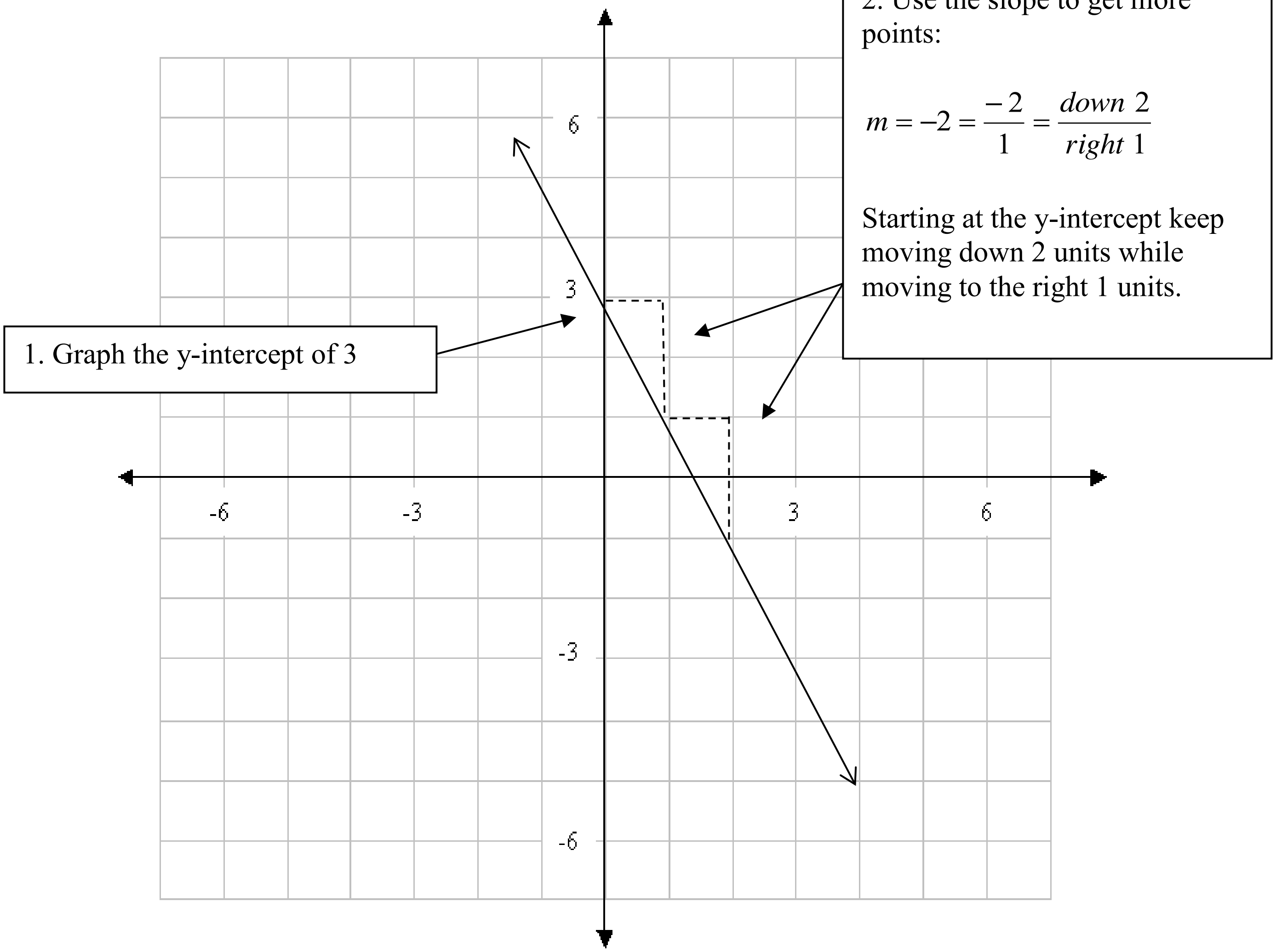
<u>Examples:</u>

11. Graph $y = \frac{1}{3}x - 2$ using the slope and the y-intercept.



Because $m = \frac{the\ change\ in\ y}{the\ change\ in\ x}$, we can read off the information for up and down from the numerator and left or right from the denominator.

12. Graph $y = -2x + 3$ using the slope and the y-intercept.



There are three ways to graph lines:

1. <u>Graphing by picking points</u>
   1. Pick three x-values. 2,0,-2 or 3,0,-3 are often good choices. Three points around x=0 that avoid fractions is best.
   2. Calculate the y-value.
   3. Graph the points on a set of axes.
   4. Draw the line. Use a straight-edge for all lines including axes.
2. <u>Graphing using the x-intercept and y-intercept:</u>
   1. Graph the x-intercept and y-intercept.
   2. Draw the line through the intercepts.
3. <u>Graphing using the slope and y-intercept.</u>
   1. Get the equation in $y = mx + b$ form to get the slope and y-intercept.
   2. Graph the y-intercept.
   3. Write the slope as a fraction and use $m = \frac{rise}{run}$ or $m = \frac{the\ change\ in\ y}{the\ change\ in\ x}$ get more points.

Exercises

1. Is $(3,5)$ a solution to $y = 2x - 1$?
2. Is $(4,1)$ a solution to $y = 3x - 11$?
3. Is $(5,-4)$ a solution to $y = -2x + 3$?
4. Is $(-3,-2)$ a solution to $y = -3x + 12$?
5. Is $(-2,3)$ a solution to $y = 2x + 7$?
6. Is $(-3,10)$ a solution to $y = -3x + 1$?
7. Is $(-2,5)$ a solution to $3x + 2y = 9$?
8. Is $(7,-6)$ a solution to $2x - 5y = 20$?

Graph by picking points:

9. $y = -2x + 3$
10. $y = \frac{1}{3}x - 2$
11. $y = -\frac{1}{3}x + 2$
12. $y = 3x - 2$
13. $y = \frac{1}{4}x + 2$
14. $y = -3x + 4$

Graph:

15. $x = 3$
16. $y = 5$
17. $y = -2$

18. $x = -3$

19. $y = -4$

20. $x = 2$

Find the x-intercept and y-intercept. Use the intercepts to graph each of the following:

21. $3x - 2y = 6$

22. $5x + 2y = -10$

23. $2x - 4y = -8$

24. $-5x + 3y = -15$

25. $-2x + 3y = 12$

26. $4x - 3y = -12$

27. $y = -3x + 6$

28. $y = \frac{1}{2}x - 2$

29. $y = -\frac{1}{3}x + 6$

30. $y = 2x - 4$

Find the slope of the line containing the two points:

31. $(3,2)$ and $(1,-6)$

32. $(2,1)$ and $(-1,-5)$

33. $(-3,5)$ and $(0,-4)$

34. $(-4,5)$ and $(2,3)$

35. $(4,3)$ and $(-2,6)$

36. $(-5,-3)$ and $(-2,9)$

37. $(3,4)$ and $(-2,4)$

38. $(2,-5)$ and $(3,-5)$

39. $(4,7)$ and $(4,2)$

40. $(3,5)$ and $(3,2)$

Put the equation in $y = mx + b$ form if it is not already and then graph by using the slope and the y-intercept.

41. $\mathrm{y} = \frac{1}{2}\mathrm{x} - 3$

42. $\mathrm{y} = \frac{1}{3}\mathrm{x} + 2$

43. $\mathrm{y} = -2\mathrm{x} + 3$

44. $\mathrm{y} = -3\mathrm{x} + 5$

45. $\mathrm{y} = -\frac{1}{3}\mathrm{x} + 2$

46. $\mathrm{y} = -\frac{1}{2}\mathrm{x} + 3$

47. $\mathrm{y} = 3\mathrm{x} - 2$

48. $\mathrm{y} = 5\mathrm{x} - 4$

49. $3\mathrm{x} - 6\mathrm{y} = 12$

50. $3\mathrm{x} - 2y = 4$

51. $2\mathrm{x} + 3\mathrm{y} = 3$

52. $2\mathrm{x} - 5\mathrm{y} = 10$

A <u>system of linear equations</u> is made up of two equations whose graphs are lines.

For instance, they may have the form $\begin{matrix} 2x + 3y = 5 \\ 5x - 4y = -3 \end{matrix}$. A point is a solution to the system of equations if it is a solution to both equations.

<u>Examples</u>

1. Is (1,2) a solution to $\begin{matrix} 2x + 3y = 8 \\ 5x - 4y = -3 \end{matrix}$?

<u>Steps</u>		<u>Reasons</u>
$2x+3y=8$ $2(1)+3(2)=8$ $2+6=8$ $8=8$	$5x-4y=-3$ $5(1)-4(2)=-3$ $5-8=-3$ $-3=-3$	Check both equations by plugging in the point.
Yes, (1,2) is a solution to the system of equations.		Since (1,2) is a solution to both equations, it is a solution to the system of equations.

Later when we learn how to solve the systems of equations, we can check our answers by plugging in our answer as above.

We will learn to solve systems of equations three ways:
1. Solve by graphing
2. Solve by substitution method
3. Solve by addition method

Remember, graphs are pictures of the solutions. Each point on a graph is a solution to the equation.

<u>To solve by graphing:</u>
1. Graph both equations on the same set of axes.
2. Where the two graphs cross (intersect) there is a solution to the system of equations.

Examples:

2. Solve by graphing: $\begin{array}{l} 3x - y = 3 \\ 2x + y = 2 \end{array}$

There are several ways to graph each line. Here we are graphing by finding the x-intercept and the y-intercept.

$3x - y = 3$

x-intercept(let y=0)

$$3x - 0 = 3$$
$$3x = 3$$
$$x = 1 \ \ or \ \ (1,0)$$

y-intercept(let x=0)

$$3(0) - y = 3$$
$$-y = 3$$
$$y = -3 \ \ or \ \ (0,-3)$$

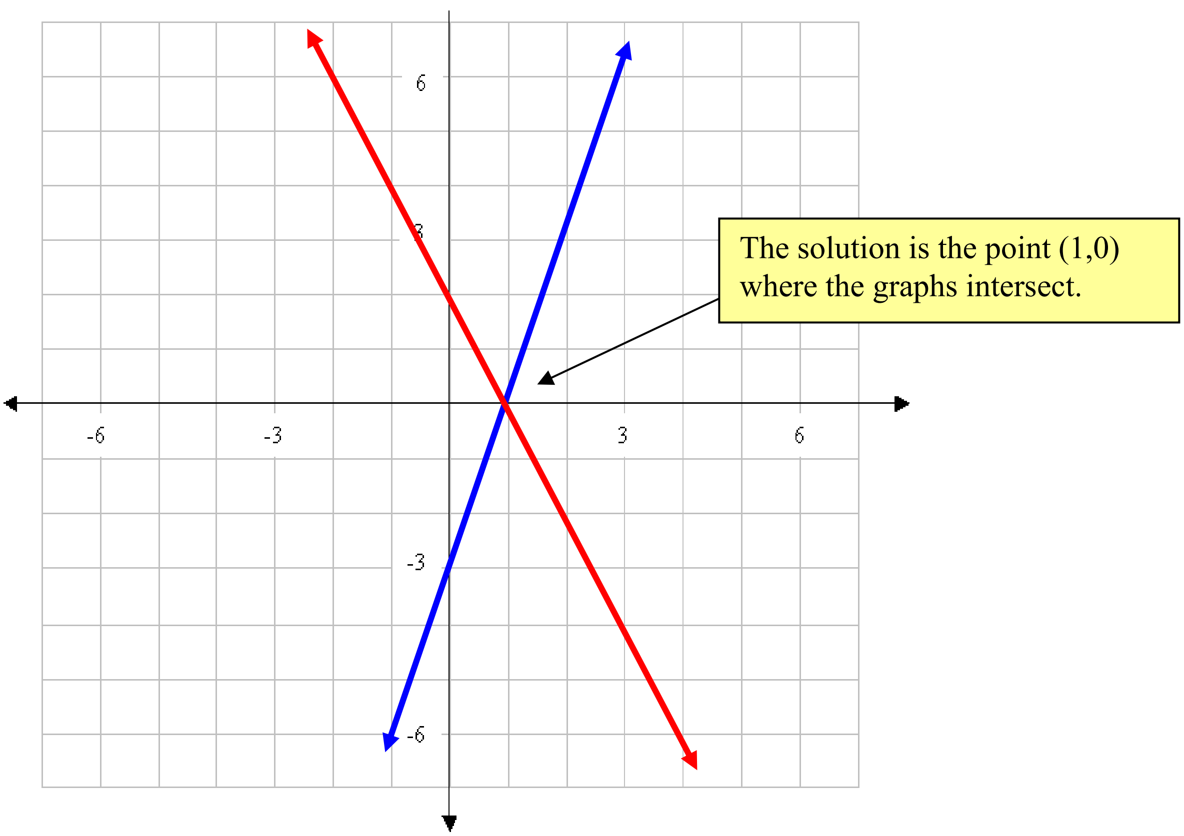
$2x + y = 2$

x-intercept(let y=0)

$$2x + y = 2$$
$$2x = 2$$
$$x = 1 \ \ or \ \ (1,0)$$

y-intercept(let x=0)

$$2(0) + y = 2$$
$$y = 2$$
$$y = 2 \ \ or \ \ (0,2)$$



3. Solve by graphing: $\begin{array}{l} x - 4 = 0 \\ y + 2 = 0 \end{array}$

Here we need to recognize that these equations represent a vertical and horizontal line.

$x - 4 = 0$

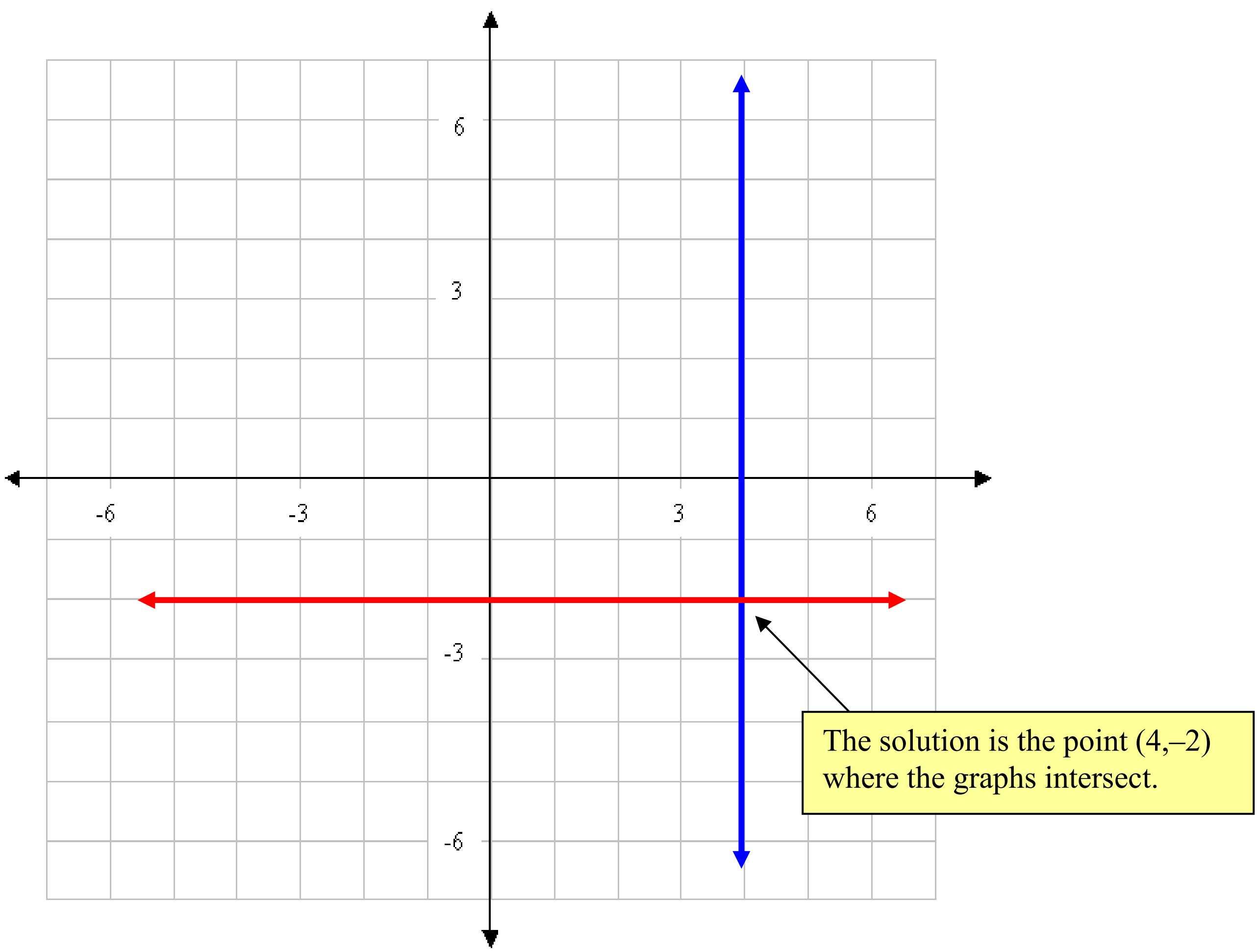
$x = 4$

Equations in this form are vertical lines (all first coordinates are 4).

$y + 2 = 0$

$y = -2$

Equations in this form are horizontal lines (all second coordinates are –2).



4. Solve by graphing: $\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 4 \end{aligned}$

There are several ways to graph each line. Below the graphs are found by finding the x-intercept and the y-intercept.

$\underline{2x - y = 6}$

x-intercept(let y=0)

$$2x - 0 = 6$$
$$2x = 6$$
$$x = 3 \;\; or \;\; (3,0)$$

y-intercept(let x=0)

$$2(0) - y = 6$$
$$-y = 6$$
$$y = -6 \;\; or \;\; (0,-6)$$

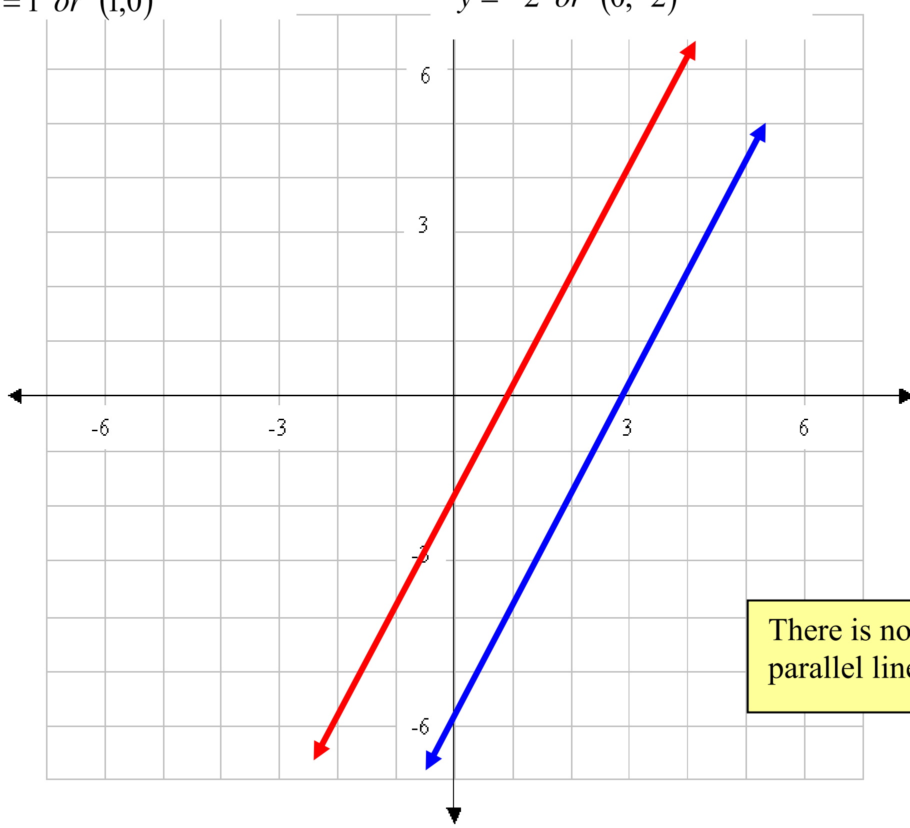
$\underline{4x - 2y = 4}$

x-intercept(let y=0)

$$4x - 2(0) = 4$$
$$4x = 4$$
$$x = 1 \;\; or \;\; (1,0)$$

y-intercept(let x=0)

$$4(0) - 2y = 4$$
$$-2y = 4$$
$$y = -2 \;\; or \;\; (0,-2)$$



There is no solution because these parallel lines never intersect.

There are three types of systems of equations. We can relate them to the three possibilities for graphing linear equations.

3 Types of Solutions:

1. **Independent System of Equations:**

1 solution $\longrightarrow$ The graphs intersect at one point.

2. **Dependent System of Equations:**

Line of solutions $\longrightarrow$ The two equations turn out to be the same equation with the same graph.

3. **Inconsistent System of Equations:**

No solutions $\longrightarrow$ The lines are parallel and never intersect.

Solving equations by graphing can be awkward, time consuming, and inexact. There are two other methods that we can use for solving systems of equations:
1. Substitution Method
2. Addition Method

Steps for solving by the substitution method:

1. Solve either equation for either variable. (Pick a variable without a number in front or you may get fractions.)
2. Substitute for the variable in the other equation.
3. Solve the resulting equation.
4. Substitute to get the other variable.
5. Check by plugging in your answer to both equations.

Examples:

5. Solve by the substitution method: $\begin{aligned} 2x + 3y &= -4 \\ x + 4y &= 3 \end{aligned}$

Steps	Reasons
$2x + 3y = -4$ $x + 4y = 3$	Solve the second equation for x. Since there is no number in front of the variable, it will be the best choice.
$x + 4y = 3$ $x = 3 - 4y$	Subtract 4y from both sides of the second equation. By solving the second equation for x, we never need to divide, which usually introduces fractions.

$2x + 3y = -4$ Replace x in the first equation with 3 – 4y. Now there is only one variable.

$2(3 - 4y) + 3y = -4$

Solve the resulting equation.

$6 - 8y + 3y = -4$

$6 - 5y = -4$

$-5y = -4 - 6$

$-5y = -10$

$\frac{-5y}{-5} = \frac{-10}{-5} \; or \; 2$

$x = 3 - 4y$ Get the other variable by replacing x in either equation.

$x = 3 - 4(2)$

$x = 3 - 8 = -5$

$(-5,2)$ or $x = -5, y = 2$ State the answer either way.

Check by replacing the variables in both equations.

$2x + 3y = -4$	$x + 4y = 3$	Both equations check.
$2(-5) + 3(2) = -4$	$-5 + 4(2) = -4$	
$-10 + 6 = -4$	$-5 + 8 = -3$	$(-5,2)$ is the solution.
$-4 = -4$	$-3 = -3$	

6. Solve by the substitution method:
$$\begin{aligned} 3x - y &= 7 \\ 2x + 3y &= 1 \end{aligned}$$

Steps	Reasons
$3x - y = 7$ $2x + 3y = 1$	Solve the first equation for y.
$3x - y = 7$ $3x - 7 = y$ $y = 3x - 7$	By solving the first equation for y, we never need to divide, which usually introduces fractions.
$2x + 3y = 1$ $2x + 3(3x - 7) = 1$	Replace y in the second equation with 3x – 7. Now there is only one variable.
$2x + 9x - 21 = 1$ $11x = 22$ $x = \frac{22}{11} \text{ or } 2$	Solve.
$y = 3x - 7$ $y = 3(2) - 7$ $y = 6 - 7 = -1$	Get the other variable by replacing x in either equation.
$(2,-1)$ or x = 2, y = –1	State your answer either way.

Check by replacing the variables in both equations.

$3x - y = 7$	$2x + 3y = 1$	Both equations check.
$3(2) - (-1) = 7$	$2(2) + 3(-1) = 1$	$(2,-1)$ is the solution.
$6 + 1 = 7$	$4 - 3 = 1$	
$7 = 0$	$1 = 1$	

7. Solve by the substitution method: $\begin{array}{l} 2x + 4y = 1 \\ x + 2y = 4 \end{array}$

Steps	Reasons
$2x + 4y = 1$ $x + 2y = 4$	Solve the second equation for x.
$x + 2y = 4$	By solving second equation for x, we never need to divide, which usually introduces fractions.
$x = -2y + 4$	
$2x + 4y = 1$	
$2(-2y + 4) + 4y = 1$	Replace x in the first equation with –2y + 4. Now there is only one variable.
$-4y + 8 + 4y = 1$	Solve.
$8 = 1$ No solution.	8 does not equal 1 regardless of what x and y are.

3 types of answers:
1. The solution is a point: **Independent systems of equations** have a one-point solution. If we graph both lines, they intersect at one point.

2. No solutions. We get a false equality. Above 8 = 1 meant there are no solutions. If we graph both lines, they are parallel and never touch. **(Inconsistent system of equations)**

3. An infinite number of solutions or a line of solutions. If we graph both lines, they are the same line. This occurs if we are left with two numbers that really are equal. If we end up with 5=5 for instance, then we can say there are an infinite number of solutions or a line of solutions. **(Dependent system of equations)**

Another method for solving systems of equations is the addition method:

Steps for solving systems of equations using the addition method:
1. Multiply one or both equations so that on of the variables will cancel after adding the two equations. (For instance 3x + (-3x) would get rid of x's.)
2. Add the equations.
3. Solve for the variable.
4. Replace the number for the variable in either equation to get the other variable.
5. State the answer.
6. Check

Examples:

8. Solve by the addition method: $\begin{aligned} 4x - 2y &= 20 \\ 5x + 2y &= 7 \end{aligned}$

Steps	Reasons
$4x - 2y = 20$ $5x + 2y = 7$	Since we have the same number with opposite signs in front of the y's, we can just add the equations.
$\begin{array}{l} 4x - 2y = 20 \\ \underline{5x + 2y = 7} \\ 9x \qquad = 27 \end{array}$	That way the y's cancel and we can easily solve for x.
$\frac{9x}{9} = \frac{27}{9}$ $x = 3$	
$5x + 2y = 7$ $5(3) + 2y = 7$ $15 + 2y = 7$ $2y = 7 - 15$ $2y = -8$ $\frac{2y}{2} = \frac{-8}{2} = -4$	Replace the x in either equation to get the other variable. Here we are using the second equation for no particular reason.
$(3,-4)$ or x=3, y= -4	State the answer either way.

Check by replacing the variables in both equations.

$$\begin{aligned} 4x - 2y &= 20 \\ 4(3) - 2(-4) &= 20 \\ 12 + 8 &= 20 \\ 20 &= 20 \end{aligned} \qquad \begin{aligned} 5x + 2y &= 7 \\ 5(3) + 2(-4) &= 7 \\ 15 - 8 &= 7 \\ 7 &= 7 \end{aligned}$$

Both equations check.

$(3,-4)$ is the solution.

9. Solve by the addition method: $\begin{aligned} 5x - 3y &= 11 \\ 2x + 4y &= -6 \end{aligned}$

Steps	Reasons
$4 \cdot (5x - 3y) = (11) \cdot 4$ $3 \cdot (2x + 4y) = (-6) \cdot 3$	Try to get the same number (but opposite sign) in front of either variable by multiplying both sides of one or both of the equations.
$20x - 12y = 44$ $6x + 12y = -18$	Here we are going to get rid of y's. Note that with the -12y and +12y we really have gotten the least common multiple of the -3 and +4 with opposite signs.
$\begin{array}{l} 20x - 12y = 44 \\ \underline{6x - 12y = -18} \\ 26x \qquad = 26 \end{array}$	Add the two equations to get rid of one of the variables.
$\dfrac{26x}{26} = \dfrac{26}{26}$	Solve for the remaining variable.
$x = 1$	
$5x - 3y = 11$ $5(1) - 3y = 11$ $-3y = 11 - 5$ $-3y = 6$ $y = -2$	Replace the number for the variable in either equation to get the other variable. Here we are using the first equation for no particular reason.
$(1, -2)$ or x=1, y= -2	State the answer either way.

Check by replacing the variables in both equations.

$5x - 3y = 11$	$2x + 4y = -6$	Both equations check.
$5(1) - 3(-2) = 11$	$2(1) + 4(-2) = -6$	$(1, -2)$ is the solution.
$5 + 6 = 11$	$2 + (-8) = -6$	
$11 = 11$	$-6 = -6$	

10. Solve by the addition method: $\begin{aligned} 3x - 6y &= 12 \\ 2x - 4y &= 8 \end{aligned}$

Steps	Reasons
$2(3x - 6y) = (12)2$ $-3(2x - 4y) = (8)(-3)$ $6x - 12y = 24$ $-6x + 12y = -24$	Try to get the same number (but opposite sign) in front of either variable by multiplying both sides of one or both of the equations. Notice that we get the least common multiple of the numbers in front of the x's with opposite signs.
$\begin{array}{r} 6x - 12y = 24 \\ -6x - 12y = -24 \\ \hline 0 \;=\; 0 \end{array}$	Add the two equations to get rid of one of the variables. As it turns out, we got rid of both variables.
There is a line of solutions. or There are an infinites number of solutions.	State your answer either way. Since zero always equals zero, the two equations represent the same line.

3 types of answers:

1. The solution is a point: **Independent systems of equations** have a one-point solution. If we graph both lines, they intersect at one point.

2. No solutions. We get a false equality. 8 = 1 meant there were no solutions. If we graph both lines, they are parallel and never touch. **(Inconsistent system of equations)**

3. An infinite number of solutions or a line of solutions. If we graph both lines, they are the same line. This occurs if we are left with two numbers that really are equal. Above we ended up with 0=0 for instance, and we can say there are an infinite number of solutions or a line of solutions. **(Dependent system of equations)**

Anytime we learn to solve a new type of equation, we are able to solve new types of application problems. There are some hints that we may be using systems of equations:
1. There are two sets of information.
2. We are looking for two unknowns

Examples:

11. Three cans of Coca Cola and one can of Mountain Dew contain 163 grams of sugar. Two cans of Coca Cola and four cans of Mountain Dew contain 262 grams of sugar. How many grams of sugar are there in each?

Let x = amount of sugar in Coca Cola and y = amount of sugar in Mountain Dew

3 cans of Coca Cola and 1 can of Mountain Dew contain 163 grams of sugar
3 x + 1 y = 163

2 cans of Coca Cola and 4 cans of Mountain Dew contain 262 grams of sugar
2 x + 4 y = 262

Solve with either method:

$$3x + y = 163$$
$$2x + 4y = 262$$

Steps	Reasons
$3x + y = 163$ $2x + 4y = 262$	Since y in the first equation does not have a number in front, it is appropriate to use the substitution method.
$3x + y = 163$ $y = 163 - 3x$	Solve the first equation for y by subtracting 3x from both sides.
$2x + 4(163 - 3x) = 262$	Substitute for y in the other equation. Solve:
$2x + 652 - 12x = 262$	Distribute the 4.
$-10x = 262 - 652$	Collect like terms and subtract 652 from both sides.
$-10x = -390$	Divide both sides by -10.
$x = 39$	Get y by substituting x into one of the equations.
$y = 163 - 3x$ $y = 163 - 3(39)$ $y = 46$	There are 39 grams of sugar in the Coca Cola and 46 grams of sugar in the can of Mountain Dew.

12. A shop has a special rate for changing all the tires on motorcycles or cars. During a week where the shop changed the tires on 27 vehicles at the special rate, 84 tires were replaced. How many motorcycles and how many cars had all the tires replaced?

Let x = number of motorcycles?
y = number of cars?

Letting variables stand for what we are trying to find is often a good trick.

"27 vehicles" means the number of motorcycles plus the number of cars equals 27.
x + y = 27

"84 tires were replaced" means:

the number of motorcycle tires plus the number of car tires equals 84.
2x + 4y = 84

Steps	Reasons
Solve: $\begin{array}{l} x+y=27 \\ 2x+4y=84 \end{array}$	Here we are solving by the addition method even though the substitution method would work as well.
$\begin{array}{l} -2(x+y)=-2(27) \\ 2x+4y=84 \end{array}$	Multiply the first equation by -2 so that we have -2x and 2x. Then adding the equation will get rid of the x's.
$\begin{array}{r} -2x+-2y=-54 \\ 2x+\ 4y=\ \ 84 \\ \hline 2y=30 \end{array}$	Solve for y.
$y=15$	
$\begin{array}{l} x+y=27 \\ x+15=27 \\ x=12 \end{array}$	Get x by replacing y in the first equation.

The shop changed the tires for 12 motorcycles and 15 cars at the special rate.

13. A broker purchased two bonds for a total of $250,000. One bond earns 7% simple annual interest, and the second one earns 8% simple annual interest. If the total annual interest from the two bonds is $18,500, what was the purchase price of each bond?

Solution
Use a chart to organize the information
Principal = amount of money invested.
Rate = annual simple interest
Interest = dollar amount that was invested
I = Prt is the formula for simple interest. We will see this in finance.

	Principal	Rate	Interest (principal X rate)
Amount at 8%	X	.08	.08x
	+		+
Amount at 7%	Y	.07	.07y
	=		=
	250,000		18,500

Since the total amount is $250,000, we can add each amount to get the total..
Since the interest adds up to $18,500 we get the equation we are need to solve.

Steps	Reasons
$x + y = 250{,}000$ $.08x + .07y = 18{,}500$	Here we are using the substitution method even though the addition method would work well.
$y = 250{,}000 - x$ $.08x + .07(250{,}000 - x) = 18{,}500$	Solve the first equation for y and substitute for y in the second equation.
$.08x + 17{,}500 - .07x = 18{,}500$ $-17{,}500 \qquad -17{,}500$	Solve the resulting equation.
$.01x = 1000$ $x = 100{,}000$	
$y = 250{,}000 - x$ $y = 250{,}000 - 100{,}000 = 150{,}000$	Get y by substituting for x in one of the equations and evaluating.

$100,000 is invested at 8% and $150,000 is invested at 7%.

14. A restaurant purchased 5 kg of tomatoes and 16 kg of potatoes for a total of $55.00. A second purchase, at the same prices, included 3 kg of tomatoes and 8 kg of potatoes for a total of $29.00. Find the cost per kilogram for the tomatoes and the potatoes.

Make two charts: one for each purchase.

First Purchase:

	Amount	Unit cost	Total
Tomatoes	5	T	5T
			+
Potatoes	16	P	16P
			=
Total			55

Second Purchase:

	Amount	Unit cost	Total
Tomatoes	3	T	3T
			+
Potatoes	8	P	8P
			=
Total			29

Solve the system of equations: $\begin{array}{l} 5T + 16P = 55 \\ 3T + 8P = 29 \end{array}$

Steps	Reasons
$5T + 16P = 55$ $3T + 8P = 29$	The two charts yield two equations if you look at the total price for each purchase.
$5T + 16P = 55$ $-2(3T + 8P) = (29)(-2)$	Here we are using the addition method.
$5T + 16P = 55$ $\underline{-6T - 16P = -58}$	The P's cancel and we solve for T.
$-T = -3$	
$T = 3$	
$3T + 8P = 29$ $3(3) + 8P = 29$	Substitute T = 3 in either equation and then solve for P
$9 + 8P = 29$	
$8P = 20$	
$P = 2.5$	

The tomatoes cost \$3 per kg and the potatoes cost \$2.50 per kg.

Exercises

1. Is $(3,1)$ a solution to
   $5x + 2y = 17$
   $3x - 2y = 7$?

2. Is $(-2,4)$ a solution to
   $3x - 2y = -14$
   $5x + y = 6$?

3. Is $(4,-1)$ a solution to
   $2x + 5y = 3$
   $4x - y = 10$?

4. Is $(3,4)$ a solution to
   $3x + 2y = 17$
   $5x - 3y = 7$?

Solve the system of equations by graphing:

5. $x + y = 4$
   $x - y = 2$

6. $2x - 3y = 6$
   $x + y = 3$

7. $y = 2x + 1$
   $y = 2x - 1$

8. $y = \frac{1}{3}x - 4$
   $y = \frac{1}{3}x + 2$

Solve by substitution:

9. $y = 2x + 3$
   $x + y = 6$

10. $y = 2x - 7$
    $x + 2y = 16$

11. $x = 3y - 4$
$2x + 3y = 1$

12. $3x - y = 12$
$x + 2y = 11$

13. $x + 2y = 8$
$2x - y = 1$

14. $3x + y = 11$
$2x - 3y = 22$

15. $x + 3y = 16$
$4x - 2y = -20$

16. $3x - y = -3$
$4x - 5y = 7$

17. $3x - 4y = -7$
$x - 4y = 3$

18. $3x - 2y = 3$
$-5x + y = 9$

Solve by the addition method:

19. $2x - y = 5$
$2x + y = 7$

20. $x + 3y = 14$
$-x + y = 2$

21. $2x + 4y = 2$
$-2x + 3y = 12$

22. $2x - 3y = 17$
$4x + 3y = 7$

23. $5x - 2y = -3$
$3x + y = 7$

24. $3x + y = -2$
    $7x + 2y = -3$

25. $2x - 3y = 7$
    $4x + 2y = 6$

26. $3x - 2y = 23$
    $x - 4y = 21$

27. $3x - 4y = 5$
    $2x + 3y = -8$

28. $5x + 3y = 7$
    $3x + 4y = 13$

29. $3x + 5y = 6$
    $2x - 3y = 23$

30. $4x + 7y = 11$
    $5x + 6y = 22$

31. $5x + 2y = -11$
    $3x - 7y = -23$

32. $-5x - 3y = -1$
    $6x + 5y = -10$

Solve the following by using an appropriate system of equations:

33. Two cans of cola and one can of root beer contain 149 grams of sugar. Three cans of cola and four cans of root beer contain 336 grams of sugar. How many grams of sugar are there in each?

34. Four cans of Fanta Orange and five cans of Sprite contain 403 grams of sugar. One can of Fanta Orange and two cans of Sprite contain 130 grams of sugar. How many grams of sugar are there in each?

35. A shop has a special rate for changing all the tires on motorcycles or cars. During a week where the shop changed the tires on 38 vehicles at the special rate, 110 tires were replaced. How many motorcycles and how many cars had all the tires replaced?

36. A shop has a special rate for changing all the tires on motorcycles or cars. During a week where the shop changed the tires on 55 vehicles at the special rate, 196 tires were replaced. How many motorcycles and how many cars had all the tires replaced?

37. A broker purchased two bonds for a total of $150,000. One bond earns 4% simple annual interest, and the second one earns 2% simple annual interest. If the total annual interest from the two bonds is $5,400, what was the purchase price of each bond?

38. A broker purchased two bonds for a total of $100,000. One bond earns 3% simple annual interest, and the second one earns 5% simple annual interest. If the total annual interest from the two bonds is $4,200, what was the purchase price of each bond?

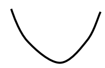
39. At a food stand outside of a European train station it is possible to buy hot dogs and beer. To purchase three hot dogs and two beers it costs 8.25 euro. For two hot dogs and three beers it costs 8.00 euro. How much does the stand charge for a hot dog? How much does it charge for a beer?

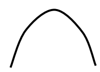
40. At Yankee Stadium in New York City, it costs $21 to purchase three beers and two hot dogs. To purchase two beers and four hot dogs, it costs $30. How much does it cost for a hot dog at Yankee Stadium? How much does it cost for a beer?

<u>Quadratic functions</u> have the form:

$f(x) = ax^2 + bx + c$ where a,b,c are real numbers

The graphs of quadratic functions are called parabolas. Their direction is determined by a:

$a > 0$ the shape is up 

$a < 0$ the shape is down 

The lowest or highest point of these parabolas is called the <u>vertex</u>. We can find the first coordinate of the vertex by:

$x = \dfrac{-b}{2a}$ for $f(x) = ax^2 + bx + c$ (the general form of the function)

Thinking of a little piece of the quadratic formula helps us remember the formula. There is a vertical line of symmetry running through the vertex.

As with lines we find:
x-intercepts by letting y = 0.
y-intercept by letting x = 0.

<u>Examples:</u>

1. For the function $f(x) = x^2 - 4x + 3$
   a. Does the parabola go up or down?
   b. Find the vertex and state the maximum or minimum.
   c. Find the x-intercepts.
   d. Find the y-intercept.
   e. Graph the function.

<u>Steps</u>	<u>Reasons</u>
$f(x) = x^2 - 4x + 3$ $a = 1, b = -4, c = 3$	
a. Since a>0, the shape is  The parabola goes up.	Examine the coefficient in front of x^2. Since there is no number in front of the x^2, the coefficient is 1 (think of $x^2 = 1x^2$).
b. Vertex $x = \dfrac{-b}{2a} = \dfrac{-(-4)}{2(1)} = \dfrac{4}{2} = 2$	The x-coordinate of the vertex is $x = \dfrac{-b}{2a}$. Replace a,b with the values from the function.
$f(2) = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$	Find the second coordinate of the vertex by evaluating the function for the x-value.
$(2,-1)$ is the vertex	

c. x-intercepts

$f(x)=x^2-4x+3$

To find the x-intercepts replace f(x) or y with zero and solve for x.

$0=x^2-4x+3$

$0=(x-3)(x-1)$

The easiest way to solve is by factoring. The factors of 3 that add up to -4 are -3 and -1: (-3)(-1)=3 and -3+(-1)=-4.

$x-3=0 \text{ or } x-1=0$

$x=3 \qquad x=1$

The x-intercepts are the points $(3,0)$ and $(1,0)$.

There are two x-intercepts. Remember the second coordinate f(x) is 0 from the start.

d. y-intercept

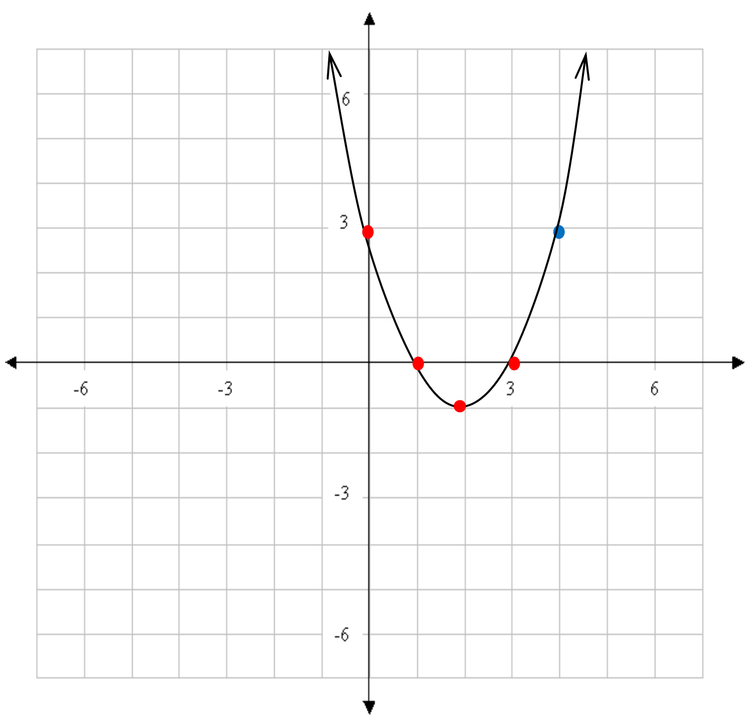
$f(0)=0^2-4(0)+3$

$=3$

The y-intercept is $(0,3)$.

To find the y-intercept let x=0. Evaluate the function for f(0).

e. Graph the points and draw the shape of a parabola. If there are not enough points to draw a parabola, then pick a few more x-values to get more points.



Graph the vertex, x-intercepts, and y-intercept (red dots). The blue dot makes sense because parabolas are symmetric.

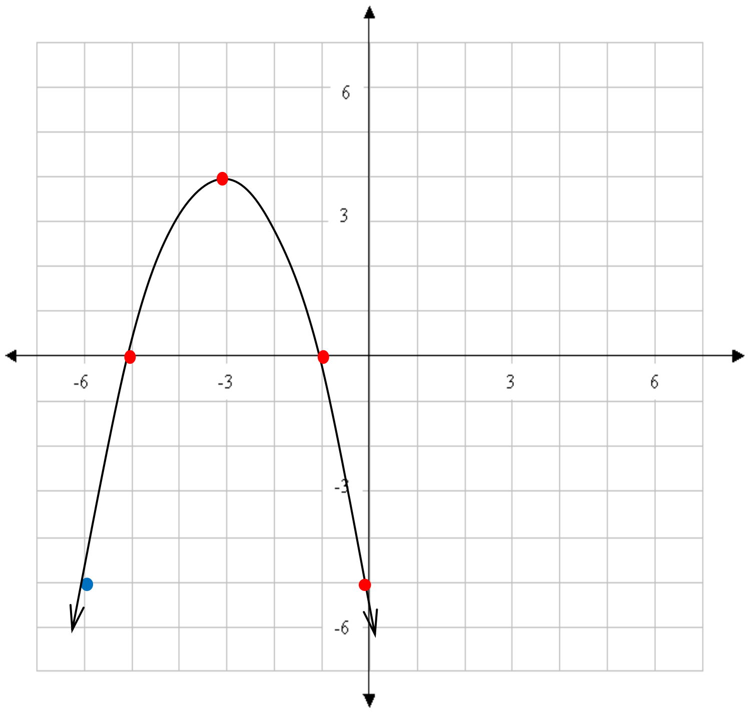
Notice that the vertex (2,-1) is the lowest point on the graph. No matter what x-value is put into the function the minimum value for f(x) is -1.

2. For the function $f(x) = -x^2 - 6x - 5$

   a. Does the parabola go up or down?
   b. Find the vertex and state the maximum or minimum.
   c. Find the x-intercepts.
   d. Find the y-intercept.
   e. Graph the function

Steps	Reasons
$f(x) = -x^2 - 6x - 5$ $a = -1, b = -6, c = -5$	Identify the coefficients (a,b,c) for the quadratic function $f(x) = ax^2 + bx + c$
a. Since a<0, the shape is $\frown$ The parabola goes down.	Examine the coefficient in front of x^2. It is -1 because $-x^2 = -1x^2$.
b. Vertex $x = \dfrac{-b}{2a} = \dfrac{-(-6)}{2(-1)} = \dfrac{6}{-2} = -3$	The x-coordinate of the vertex is $x = \dfrac{-b}{2a}$.
$f(-3) = -(-3)^2 - 6(-3) - 5 = -9 + 18 - 5 = 4$ $(-3,4)$ is the vertex.	Find the second coordinate of the vertex by evaluating the function for the x-value.
c. x-intercepts $f(x) = -x^2 - 6x - 5$	To find the x-intercepts replace f(x) or y with zero and solve for x.
$0 = -x^2 - 6x - 5$ $-1(0) = -1(-x^2 - 6x - 5)$ $0 = x^2 + 6x + 5$	Before trying to solve by factoring it is very helpful to multiply both sides by -1. Making sure that the x^2 term is positive makes the factoring much easier.
$0 = (x+5)(x+1)$	The factors of 5 that add up to 6 are 5 and 1. (5)(1)=5 and 5+1=6.
$x + 5 = 0 \ \ or \ \ x + 1 = 0$ $x = -5 \qquad\qquad x = -1$	
The x-intercepts are the points $(-5,0)$ and $(-1,0)$.	There are two x-intercepts. Remember the second coordinate f(x) is 0.
d. y-intercept $f(0) = -0^2 - 6(0) - 5 = -5$ The y-intercept is $(0,-5)$.	To find the y-intercept let x=0. Evaluate the function for f(0).

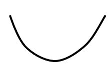
e. Graph the points and draw the shape of a parabola. If there are not enough points to draw a parabola, then pick a few more x-values to get more points.



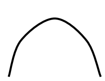
Graph the vertex, x-intercepts, and y-intercept (red dots). The blue dot makes sense because parabolas are symmetric.

Notice that the vertex (-3,4) is the highest point on the graph. That means that regardless of the x-values the maximum value for f(x) is 4.

Remember the shape of the parabola is determined by the a:

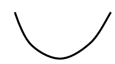
$a > 0$ the shape is up 

Here there is a lowest point or minimum. <u>The minimum of the quadratic function</u> is the second coordinate of the vertex when $a > 0$

$a < 0$ the shape is down 
Here there is a highest point or maximum. <u>The maximum of the quadratic function</u> is the second coordinate of the vertex when $a < 0$.

Examples:

3. Find the maximum or minimum of the function $f(x)=2x^2-12x+30$

Steps	Reasons
$f(x)=2x^2-12x+30$	Find the vertex:
$x=\dfrac{-b}{2a}$	1. Use $x=\dfrac{-b}{2a}$ to find the x-coordinate of the vertex.
$x=\dfrac{-(-12)}{2(2)}=\dfrac{12}{4}=3$	
$f(3)=2(3)^2-12(3)+30$ $=2\cdot 9-36+30$ $=12$	2. Replace x with 3 to get the second coordinate.
The minimum is 12	Because $a>0$ the shape is up  So, there is a lowest point or minimum. The minimum of the quadratic function is the second coordinate of the vertex when $a>0$

We can use this notion of finding maximums and minimums to solve important word problems. In business we want to maximize profit and minimize cost.

4. A manufacturer believes that the profit in dollars, P, the company makes is related to the number of coffee makers produced and sold, x, by the function $P(x)=-0.03x^2+60x-7000$. What is the maximum profit that the manufacturer can expect?

Steps	Reasons
$P(x)=-0.03x^2+60x-7000$	Find the vertex:
$x=\dfrac{-b}{2a}=\dfrac{-60}{2(-0.03)}=\dfrac{-60}{-0.06}=1000$	Use $x=\dfrac{-b}{2a}$ to find the x-coordinate of the vertex.
$P(1000)=-0.03(1000)^2+60(1000)-7000$ $=-0.03\cdot 1{,}000{,}000+60{,}000-7000$ $=-30{,}000+60{,}000-7000$ $=23{,}000$	Replace x with 1000 to get the second coordinate.
The maximum profit of \$23,000 occurs when 1000 coffee makers are produced and sold.	Write the answer.

Exercises

Answer a-e for the following quadratic functions:

a. Does the parabola go up or down?
b. Find the vertex and state the maximum or minimum.
c. Find the x-intercepts.
d. Find the y-intercept.
e. Graph the function

1. $f(x) = x^2 + 4x + 3$

2. $f(x) = x^2 + 6x + 5$

3. $f(x) = x^2 + 2x - 3$

4. $f(x) = -x^2 + 2x + 3$

5. $f(x) = -x^2 + 4x - 3$

6. $f(x) = 2x^2 - 4x - 6$

7. $f(x) = 2x^2 - 8x + 6$

8. $f(x) = -2x^2 - 4x + 6$

9. $f(x) = -2x^2 + 4x + 6$

10. $f(x) = -2x^2 + 12x - 10$

Find the maximum or minimum of the function and say whether it is a maximum or minimum.

11. $f(x) = 3x^2 + 18x - 7$

12. $f(x) = 5x^2 - 20x + 11$

13. $f(x) = -6x^2 - 24x + 18$

14. $f(x) = -7x^2 + 28x + 15$

15. $f(x) = 15x^2 - 120x + 100$

16. $f(x) = -20x^2 + 2000x - 10{,}000$

17. $f(x) = -100x^2 + 5000x + 20{,}000$

18. $f(x) = 80x^2 - 4800x + 15{,}000$

Solve the following application problems:

19. A company that installs security systems believes that the profit in dollars, P, the company makes is related to the number of security systems installed, x, by the function $P(x) = -0.04x^2 + 1600x - 12{,}000$. What is the maximum profit that the company can expect?

20. A company that rents portable toilets believes that the profit in dollars, P, the company makes is related to the number of portable toilets rented, x, by the function $P(x) = -0.1x^2 + 500x - 18{,}500$. What is the maximum profit that the company can expect?

21. An online clothing company is deciding whether or not to sell a certain type of boot. The company believes that the profit in dollars, P, is related to the number of boots produced and sold, x, by the function $P(x) = -0.4x^2 + 240x - 1500$. What is the maximum profit that the company can expect?

22. An object launched straight up at a speed of 14.7 meters per second has a height, h, in meters of $h(t) = -4.9t^2 + 14.7t$, t seconds after the object is launched. What is the maximum height that the object will reach? Why does t have to be between zero and three? (This problem does not take into account air resistance.)

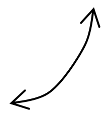
23. An object launched straight up at a speed of 29.4 meters per second has a height, h, in meters of $h(t) = -4.9t^2 + 29.4t$, t seconds after the object is launched. What is the maximum height that the object will reach? Why does t have to be between zero and six? (This problem does not take into account air resistance.)

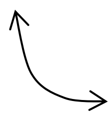
24. A .44 magnum bullet leaves the barrel of the gun at about 392 meters per second. If a .44 magnum bullet is shot straight up in the air, it has a height, h, in meters of $h(t) = -4.9t^2 + 392t$, t seconds after the bullet leaves the barrel. What is the maximum height that the object will reach? (This problem does not take into account air resistance. The actual height of the .44 magnum bullet will be much less than the value we found because of air resistance.)

An <u>exponential function</u> has the form:

$f(x) = b^x$, b is called the base and b is a positive real number.
Also, b cannot be equal to one.

When we graph an exponential function, we get two main types of graphs.

1. $b > 1$ 

2. $0 < b < 1$ 

We can draw graphs of $f(x) = b^x$ by picking points until we get the above shapes.

<u>Examples:</u>

1. Graph $f(x) = 2^x$

Pick some first coordinates and calculate the second coordinates using the function: $f(x) = 2^x$ using -3, -2, -1, 0, 1, 2, 3 will usually be a good start.

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$
$$f(0) = 2^0 = 1$$
$$f(1) = 2^1 = 2$$
$$f(2) = 2^2 = 4$$
$$f(3) = 2^3 = 8$$

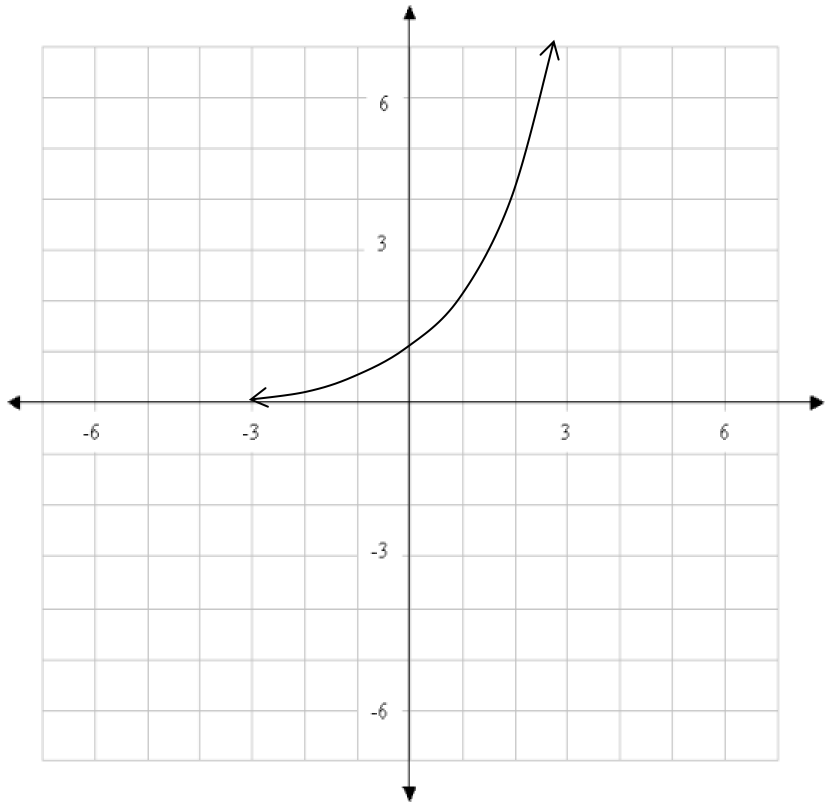
Remember how to evaluate negative exponents:

$$b^{-m} = \frac{1}{b^m}$$

$$b^0 = 1$$

Note the following:
1. The graph is increasing and has the same shape as $b > 1$ from above.
2. On the left the graph gets closer to the x-axis without touching.

$f(x)=2^x$



2. Graph $f(x)=3^x$

Pick some first coordinates and calculate the second coordinates using the function: $f(x)=3^x$

x	$f(x)$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

$f(-2)=3^{-2}=\frac{1}{3^2}=\frac{1}{9}$

$f(-1)=3^{-1}=\frac{1}{3^1}=\frac{1}{3}$

$f(0)=3^0=1$

$f(1)=3^1=3$

$f(2)=3^2=9$

Remember how to evaluate negative exponents:

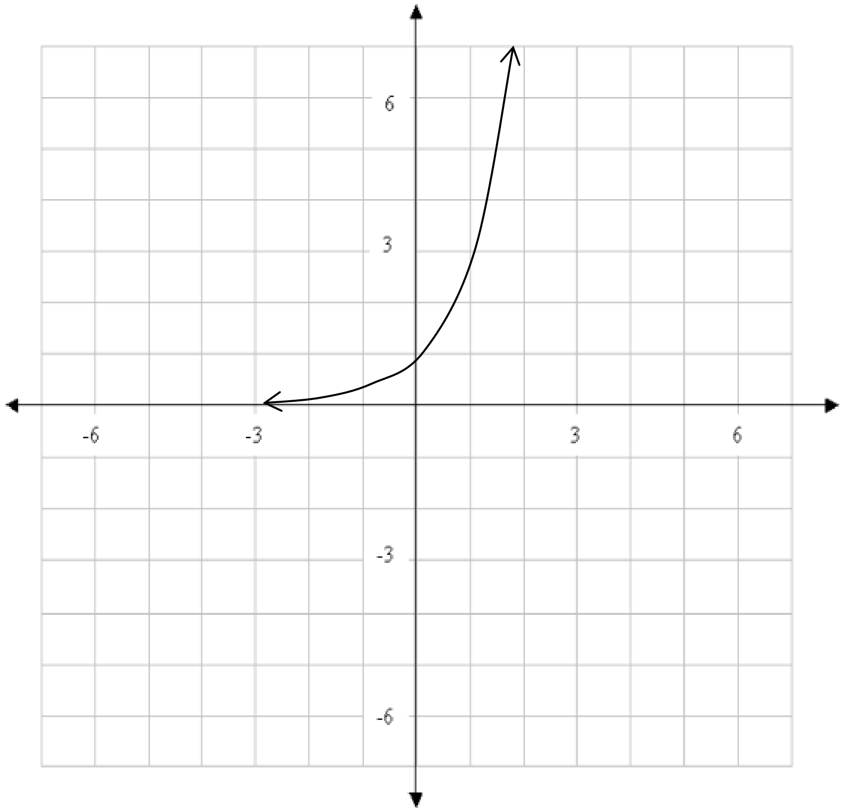
$b^{-m}=\frac{1}{b^m}$

$b^0=1$

Note the following:

1. The graph is increasing and has the same shape as $b>1$ from above.
2. On the left the graph gets closer to the x-axis without touching.

$f(x) = 3^x$



3. Graph $f(x) = \left(\frac{1}{2}\right)^x$

Pick some first coordinates and calculate the second coordinates:

$f(x) = \left(\frac{1}{2}\right)^x$

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$$f(-3) = \left(\frac{1}{2}\right)^{-3} = 2^3 = 8$$

$$f(-2) = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$$

$$f(-1) = \left(\frac{1}{2}\right)^{-1} = 2^1 = 2$$

$$f(0) = \left(\frac{1}{2}\right)^{0} = 2^0 = 1$$

$$f(1) = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$$

$$f(2) = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$$

To evaluate negative exponents:

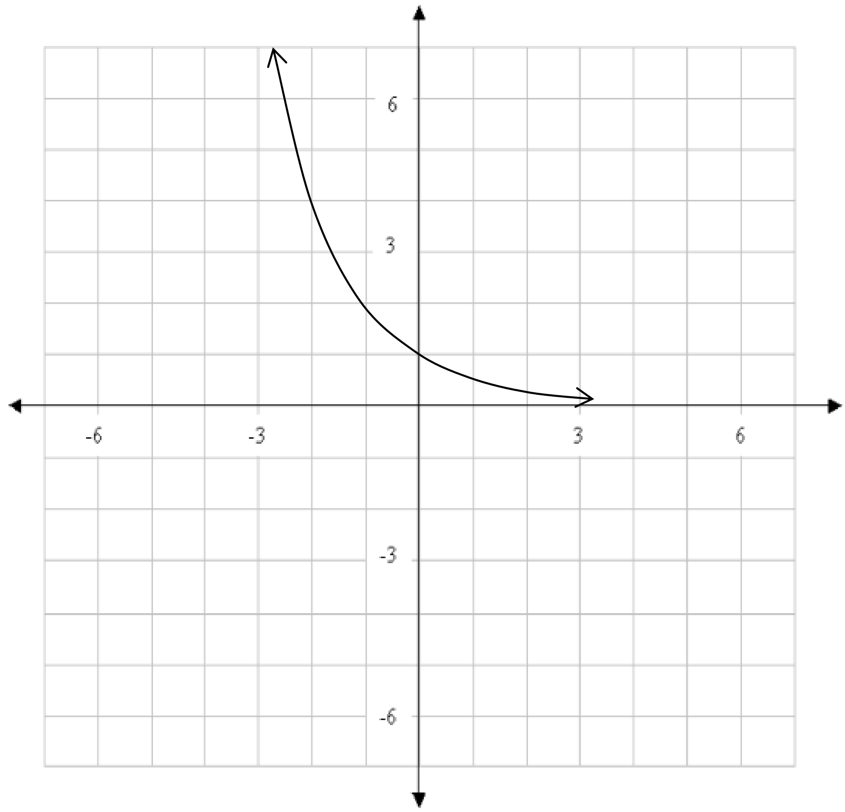
$b^{-m} = \frac{1}{b^m}$ or $\frac{1}{b^{-m}} = b^m$

$b^0 = 1$

Note the following:

1. The graph is decreasing and has the same shape as $0<b<1$ from above.
2. On the right the graph gets closer to the x-axis without touching.

$f(x)=\left(\frac{1}{2}\right)^x$



4. Graph: $f(x)=2^{x-3}$

x	$f(x)$
1	$\frac{1}{4}$
2	$\frac{1}{2}$
3	1
4	2
5	4
6	8

$f(1)=2^{1-3}=2^{-2}=\frac{1}{2^2}=\frac{1}{4}$

$f(2)=2^{2-3}=2^{-1}=\frac{1}{2^1}=\frac{1}{2}$

$f(3)=2^{3-3}=2^0=1$

$f(4)=2^{4-3}=2^1=2$

$f(5)=2^{5-3}=2^2=4$

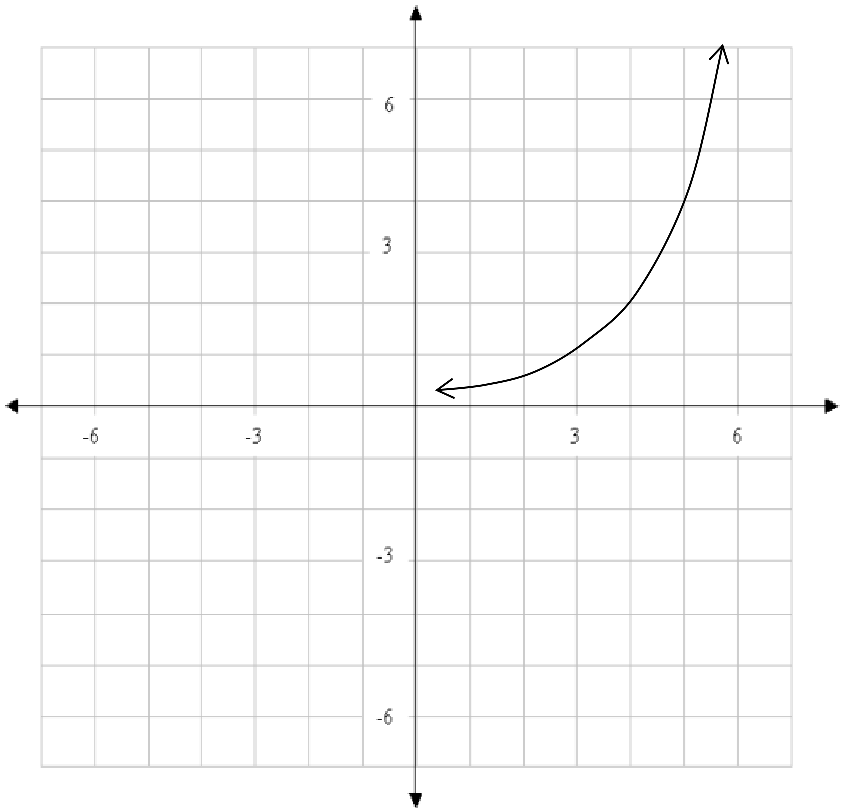
$f(6)=2^{6-3}=2^3=8$

To evaluate negative exponents:

$b^{-m}=\frac{1}{b^m}$

$b^0=1$

$f(x) = 2^{x-3}$



Note that this is really the same graph as $f(x) = 2^x$ moved one unit to the right.

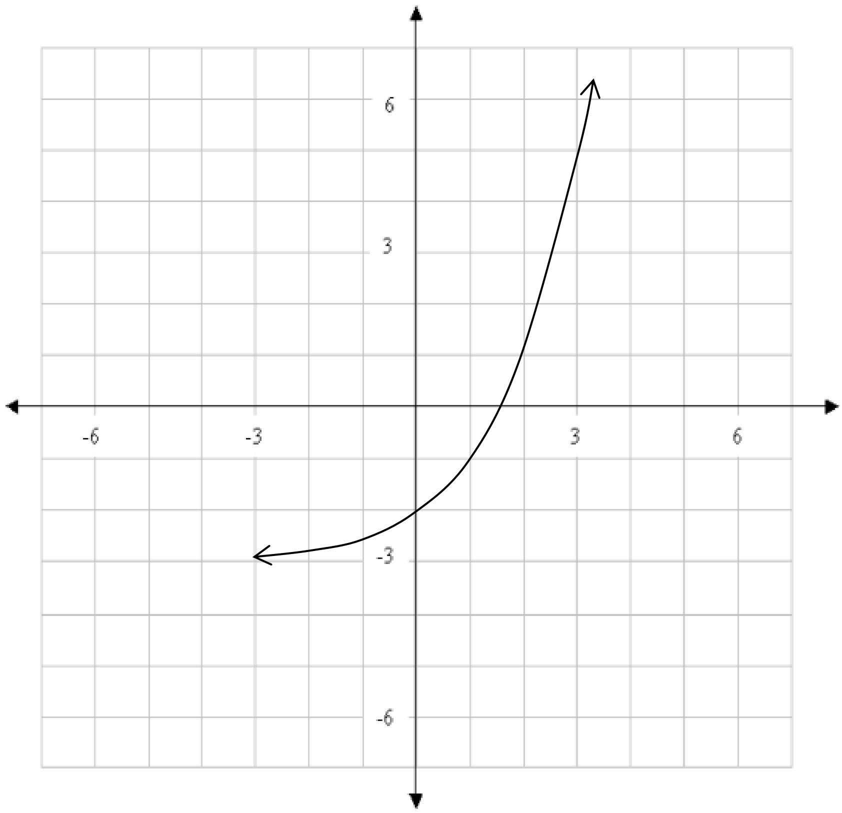
5. Graph: $f(x) = 2^x - 3$

x	$f(x)$
-2	-2.75
-1	-2.5
0	-2
1	-1
2	1
3	5

$$f(-2) = 2^{-2} - 3 = \frac{1}{2^2} - 3 = \frac{1}{4} - 3 = -2.75$$
$$f(-1) = 2^{-1} - 3 = \frac{1}{2} - 3 = -2.5$$
$$f(0) = 2^0 - 3 = 1 - 3 = -2$$
$$f(1) = 2^1 - 3 = 2 - 3 = -1$$
$$f(2) = 2^2 - 3 = 4 - 3 = 1$$
$$f(3) = 2^3 - 3 = 8 - 3 = 5$$

Note that the graph below is really the same graph as $f(x) = 2^x$ moved down three units.

$f(x)=2^x-3$



6. Graph $f(x)=2^{-x}$

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

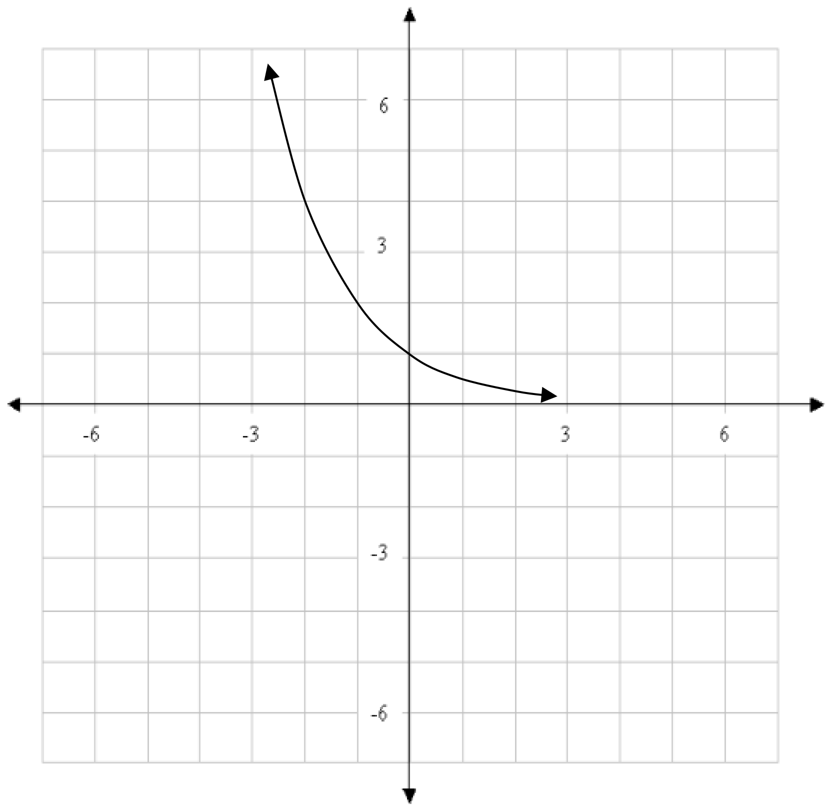
$$f(-3)=2^{-(-3)}=2^3=8$$
$$f(-2)=2^{-(-2)}=2^2=4$$
$$f(-1)=2^{-(-1)}=2^1=2$$
$$f(0)=2^{-0}=2^0=1$$
$$f(1)=2^{-1}=\frac{1}{2}$$
$$f(2)=2^{-2}=\frac{1}{2^2}=\frac{1}{4}$$

Note the following:

1. The graph is decreasing and has the same shape as $0<b<1$ from above.
2. On the right the graph gets closer to the x-axis without touching.

The graphs of $f(x)=\left(\frac{1}{2}\right)^x$ and $f(x)=2^{-x}$ are the same because $2^{-1}=\frac{1}{2}$

$f(x)=2^{-x}$



Calculators:

We can use our calculators to evaluate exponents. Everybody should have a basic scientific calculator with buttons like

- x^y or y^x or $\wedge$
- ln
- log

To find 3^2 on your calculator, usually you have to use one of the following two orders on your calculator. When you get 9, those are the steps that you should follow.

1. Push 3. 2. Push x^y or y^x or $\wedge$. You should have only one. 3. Push 2 4. Push =

or (depending on your calculator)

1. Push 2. 2. Push x^y or y^x or $\wedge$. You should have only one. 3. Push 3 4. Push =

We have a new number "e". It is used in both decay and growth problems. "e" is an irrational number like π. So, we cannot write it as a decimal number, but it is on most calculators. "e" is about 2.71828…

Find e^3 (answer: 20.0855269…)

1. INV, Shift, or 2^{nd}. You should have only one.
2. Push ln.
3. Push 3
4. Push =

or (depending on your calculator)

1. Push 3
2. INV, Shift, or 2^{nd}. You should have only one.
3. Push ln.
4. Push =

Above the "ln" button you usually see the e^x button, which is what we are really using.

Find e^{-4} (answer: 0.0183156…)

1. INV, Shift, or 2^{nd}. You should have only one.
2. Push ln.
3. Push $(-)$ or $\pm$. You should have only one.
4. Push 3
5. Push =

or (depending on your calculator)

1. Push 3
2. Push $(-)$ or $\pm$. You should have only one
3. INV, Shift, or 2^{nd}. You should have only one.
4. Push ln.
5. Push =

Examples:

7. Doubling time is the time it takes for something to double in size, number, or amount. If the world population was about 3 billion in 1960 and 6 billion in 2000 then the doubling time is 40 years. If the world population, $\mathrm{P(t)}$, measured in billions follows the function $\mathrm{P(t)} = 6 \cdot 2^{0.025\,\mathrm{t}}$ where *t* is the number of years after 2000, what will the world population be in the year 2200?

Solution:

t = 200 because 2200 is 200 years after 2000
Find *P*(*t*), by plugging *t* into the function

$\mathrm{P(t)} = 6 \cdot 2^{0.025\,\mathrm{t}}$
$\mathrm{P(200)} = 6 \cdot 2^{0.025\,(200)} = 192$

In the year 2200 the world's population will be 192 billion if the doubling time remains the same.

8. Carbon-14 is found in all living organisms. When a living organism dies, the carbon-14 begins to slowly decay and become nitrogen-14. We can determine the time since an organism was living by measuring the amount of carbon-14 that remains. The half-life of carbon-14 is about 5730 years, which means that half of the carbon-14 remains 5730 years after a living organism dies.

A living organism that has 50 grams of carbon-14 will have $\mathrm{N(t)} = 50\mathrm{e}^{-0.000121\,\mathrm{t}}$ grams of carbon-14 *t* years after it dies. How much carbon-14 will remain 10,000 years after the organism dies?

Solution:
t years after the organism dies
$\mathrm{N(t)}$ is the grams of carbon-14

For t = $10{,}000$ find $\mathrm{N(t)}$
$\mathrm{N(10{,}000)} = 50\mathrm{e}^{-0.000121\,(10{,}000)} = 50\mathrm{e}^{-1.21} = 14.90986\ldots$

About 14.9 grams of carbon-14 will remain after 10,000 years.

Exercises

Graph the following on graph paper using a straight-edge for the axes. Remember that graph paper can be downloaded from the Internet easily.

1. $\mathrm{f(x)} = 3^{\mathrm{x}}$

2. $\mathrm{f(x)} = 4^{\mathrm{x}}$

3. $\mathrm{f(x)} = \left(\frac{1}{3}\right)^{\mathrm{x}}$

4. $\mathrm{f(x)} = \left(\frac{1}{4}\right)^{\mathrm{x}}$

5. $\mathrm{f(x)} = 2^{\mathrm{x}+1}$

6. $\mathrm{f(x)} = 2^{\mathrm{x}-2}$

7. $\mathrm{f(x)} = 3^{\mathrm{x}-3}$

8. $\mathrm{f(x)} = 3^{\mathrm{x}+2}$

9. $\mathrm{f(x)} = 2^{\mathrm{x}} - 5$

10. $\mathrm{f(x)} = 2^{\mathrm{x}} + 1$

11. $\mathrm{f(x)} = 3^{\mathrm{x}} - 4$

12. $\mathrm{f(x)} = 3^{\mathrm{x}} + 2$

13. $\mathrm{f(x)} = 3^{-\mathrm{x}}$

14. $\mathrm{f(x)} = 4^{-\mathrm{x}}$

15. Doubling time is the time it takes for something to double in size, number, or amount. The number of people with a disease may double over a few months, years, or even decades. In a particular country if the number of people with malaria, $N(t)$, measured in millions follows the function $N(t) = 5 \cdot 2^{0.04\,t}$ where t is the number of years after 1960, how many people had malaria in 1960? In

1985? How long was the doubling time for the number of people with malaria according to the formula?

16. Doubling time is the time it takes for something to double in size, number, or amount. For instance the number of rabbits in an area may double quickly under ideal circumstances. If the number of rabbits in area, $\mathrm{N(t)}$, follows the function
$\mathrm{N(t)} = 100 \cdot 2^{0.125\,\mathrm{t}}$ where t is the number of weeks after some start date. How many rabbits are there 8 weeks later? How many rabbits are there 16 weeks later? How long was the doubling time for the number of rabbits according to the formula?

For the next two problems we will consider carbon-14, which is found in all living organisms. When a living organism dies, the carbon-14 begins to slowly decay and become nitrogen-14. We can determine the time since an organism was living by measuring the amount of carbon-14 that remains. The half-life of carbon-14 is about 5730 years, which means that half of the carbon-14 remains 5730 years after a living organism dies.

17. A living organism that has 50 grams of carbon-14 will have $\mathrm{N(t)} = 50\mathrm{e}^{-0.000121\cdot\mathrm{t}}$ grams of carbon-14 t years after it dies. How much carbon-14 will remain 10,000 years after the organism dies?

18. A living organism that has 80 grams of carbon-14 will have $\mathrm{N(t)} = 80\mathrm{e}^{-0.000121\cdot\mathrm{t}}$ grams of carbon-14 t years after it dies. How much carbon-14 will remain 7000 years after the organism dies?

Definition of a logarithm:

$y = \log_b x$ means $x = b^y$

x > 0, b > 0, b≠1

Note that we just switched the x and y in the exponential relation. We are able to go back and forth writing either the logarithmic form or exponential form.

Examples:

1. Write $81 = 3^4$ in logarithmic form.

Steps	Reasons
$81 = 3^4$	$y = \log_b x$ means $x = b^y$ I focus on:
$= \log$	1. The bases are the same.
$= \log_3$	2. The number by itself for the logarithm is the exponent in exponential form.
$4 = \log_3$	3. The last number goes into the only remaining position.
$4 = \log_3 81$	You can do this showing only the last step.

2. Write $2^{-3} = \frac{1}{8}$ in logarithmic form.

Steps	Reasons
$2^{-3} = \frac{1}{8}$	$y = \log_b x$ means $x = b^y$ I focus on:
$= \log$	The bases are the same.
$= \log_2$	The exponent in exponential form is the number by itself for the logarithm.
$-3 = \log_2$	The last number goes into the only remaining position.
$-3 = \log_2 \frac{1}{8}$	You can do this showing only the last step.

3. Write the $\log_5 \frac{1}{25} = -2$ in exponential form

Steps	Reasons
$\log_5 \frac{1}{25} = -2$	The bases are the same.
$= 5$	The number by itself for the logarithm is the exponent in exponential form.
$= 5^{-2}$	The last number goes into the only remaining position.
$\frac{1}{25} = 5^{-2}$	You can do this showing only the last step.

There are some special logarithms where the base is not written, but understood. These logarithms are on scientific calculators.

The common logarithm is base 10:
$\log x = \log_{10} x$ If no base is written, then it is base 10.

The natural logarithm is base e:
$\ln x = \log_e x$ If ln is written, then it is base e. $e \approx 2.7.1828...$ Since e is an irrational number, we cannot write it out as a decimal number without rounding off. The number e comes by looking at continuous growth and decay.

Steps for evaluating logarithms:
1. Let y = the logarithm
2. Rewrite as an exponential relation.
3. Determine what y should be.
4. The y will be the answer.

Examples:
4. Evaluate $\log 1000$.

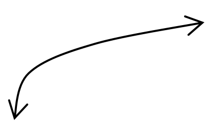
Steps	Reasons
$\log 1000$	Let y = the logarithm. If no base is written, then it is base 10.
$y = \log 1000$	
$y = \log_{10} 1000$	Rewrite as an exponential relation using
$1000 = 10^y$	$y = \log_b x$ means $x = b^y$
$y = 3$	Determine what y should be. The y is the answer.
$\log 1000 = 3$	Write the answer.

5. Evaluate $\ln e^4$.

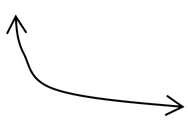
Steps	Reasons
$\ln e^4$	
	Let y = the logarithm.
$y = \ln e^4$	
	If ln is written, then it is base e.
$y = \log_e e^4$	
$e^4 = e^y$	Rewrite as an exponential relation using $y = \log_b x$ means $x = b^y$
$y = 4$	Determine what y should be. The y is the answer.
$\ln e^4 = 4$	Write the answer.

All logarithmic functions have one of the following two shapes depending on whether the base is greater than one or between zero and one.

$y = \log_b x$ for b > 1 has the shape:



$y = \log_b x$ for 0 < b < 1 has the shape:



Steps for graphing logarithmic functions:

1. Replace $f(x)$ with y
2. Rewrite the logarithmic equation as an exponential equation using the definition: $y = \log_b x$ means $x = b^y$
3. Pick some y values (-2,-1,0,1,2 for instance) and calculate the x coordinate.
4. Graph the points and draw the graph as above. More points may be needed if the shape is not clear.

Examples:

6. Graph: $f(x)=\log_3 x$

1. Replace $f(x)$ with y

$y=\log_3 x$

2. Rewrite the logarithmic equation as an exponential equation using the definition: $y=\log_b x$ means $x=b^y$

$x=3^y$

3. Pick some y values (-2,-1,0,1,2 for instance) and calculate the x coordinate.

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

$3^{-2}=\frac{1}{3^2}=\frac{1}{9}$

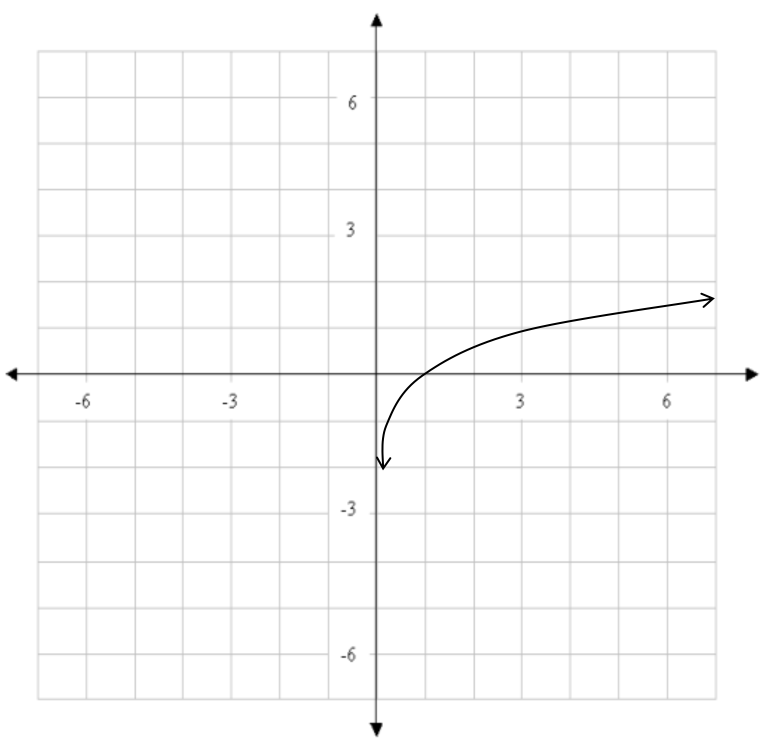
$3^{-1}=\frac{1}{3^1}=\frac{1}{3}$

$3^0=1$

$3^1=3$

$3^2=9$

Pick y-values and calculate x-values. Be careful to keep the x's in the first column and y's in the second.



Notice that the graph approaches a vertical line on the left.

7. Graph: $f(x)=\log_3(x+2)$

$y=\log_3(x+2)$

1. Replace $f(x)$ with y
2. Rewrite the logarithmic equation as an exponential equation using the definition: $y=\log_b x$ means $x=b^y$
3. Pick some y values (-2,-1,0,1,2 for instance) and calculate the x coordinate.

$x+2=3^y$

$x=3^y-2$

x	y
$-1\frac{8}{9}$	-2
$-1\frac{2}{3}$	-1
-1	0
1	1
7	2

$3^{-2}-2=\frac{1}{3^2}-2=\frac{1}{9}-2=-1\frac{8}{9}$

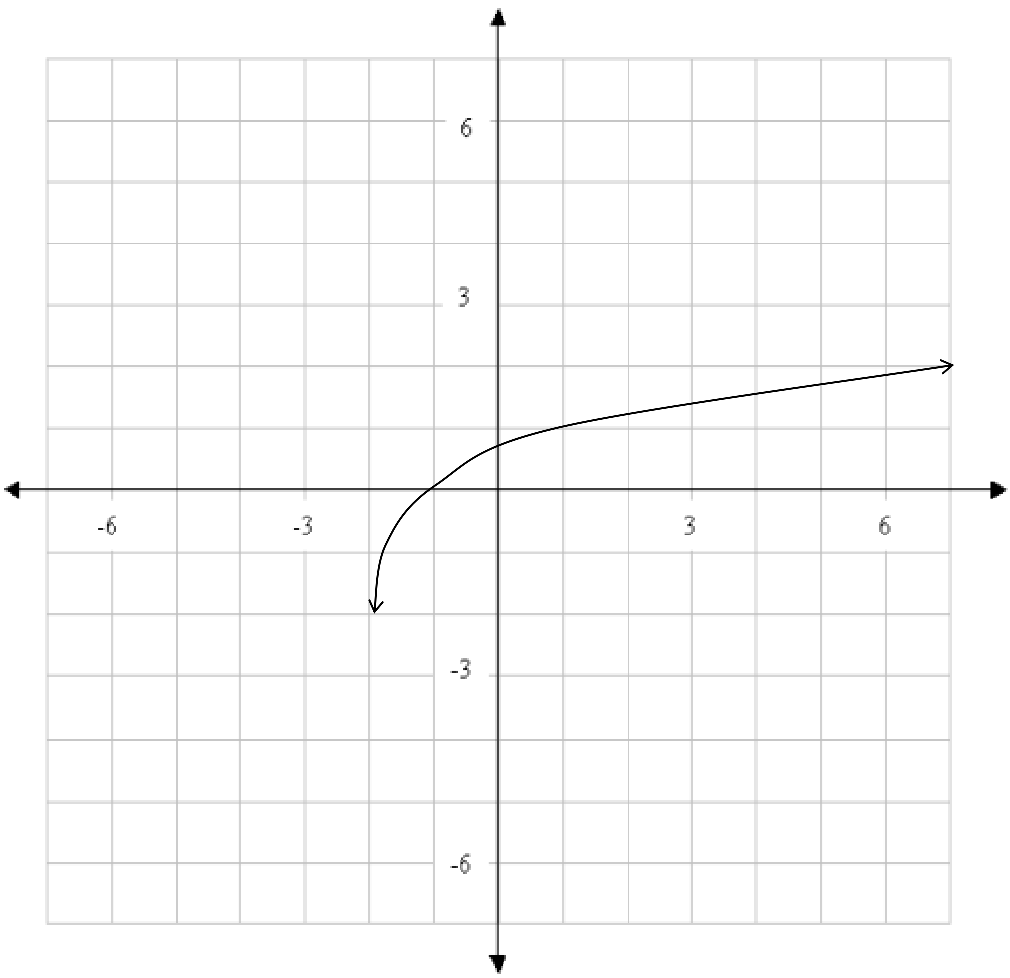
$3^{-1}-2=\frac{1}{3^1}-2=\frac{1}{3}-2=-1\frac{2}{3}$

$3^0-2=1-2=-1$

$3^1-2=3-2=1$

$3^2-2=9-2=7$

Pick y-values and calculate x-values. Be careful to keep the x's in the first column and y's in the second.



Notice that the graph approaches a vertical line on the left.

8. Graph: $f(x)=\log_{\frac{1}{2}} x$

$y=\log_{\frac{1}{2}} x$

$x=\left(\frac{1}{2}\right)^{y}$

x	y
$\frac{1}{8}$	3
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3

1. Replace $f(x)$ with y

2. Rewrite the logarithmic equation as an exponential equation using the definition: $y=\log_b x$

3. Pick some y values (-3,-2,-1,0,1,2,3 coordinate.

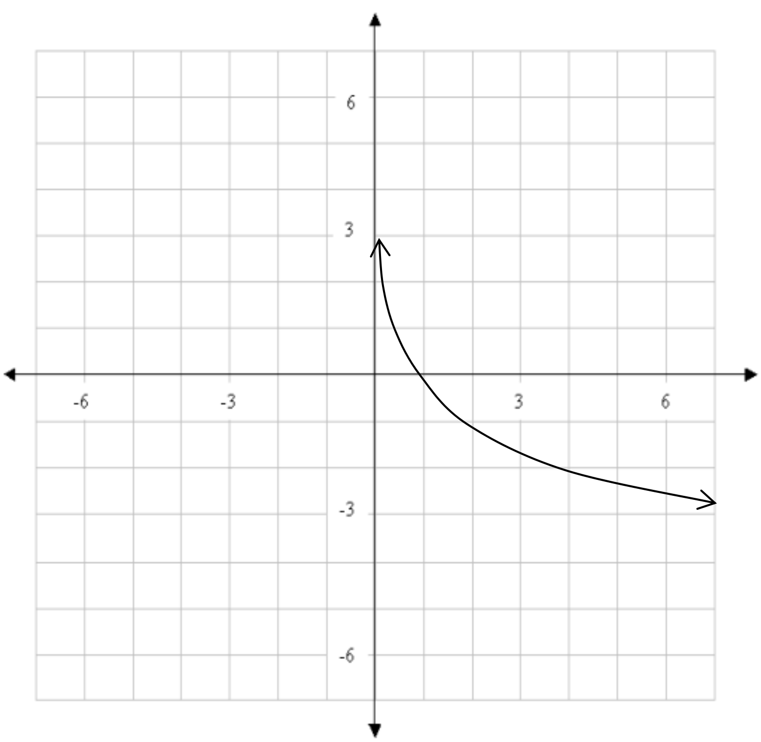
Pick y-values and calculate x-values. Be careful to keep the x's in the first column and y's in the second.

$\left(\frac{1}{2}\right)^{-3}=2^{3}=8$ $\quad$ $\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$

$\left(\frac{1}{2}\right)^{-2}=2^{2}=4$ $\quad$ $\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$

$\left(\frac{1}{2}\right)^{-1}=2^{1}=2$ $\quad$ $\left(\frac{1}{2}\right)^{3}=\frac{1}{8}$

$\left(\frac{1}{2}\right)^{0}=1$



$f(x)=\log_{\frac{1}{2}} x$

9. Advertising dollars are well spent up to a point. For instance a well-known company may decide to spend another ten million dollars, but it is unlikely that they will reach people that have not heard of their product. Suppose the function $\mathrm{f(x)} = 10{,}000 \cdot \log(\mathrm{x})$ models the number of people that will hear of a new product for x equal to the amount spent on advertising in dollars. How many people will have heard of the new product after \$500,000 is spent on advertising? After \$1,000,000 is spent? Is it worth spending the extra money on advertising?

Steps	Reasons
$\mathrm{f(x)} = 10{,}000 \cdot \log(x)$	Plug x = 500,000 into the function.
$\mathrm{f}(500{,}000) = 10{,}000 \cdot \log(500{,}000)$	Use a scientific calculator with a "log"
$\mathrm{f}(500{,}000) = 56{,}989.700\ldots$	button, which is log base 10.

About 57,000 people will have heard of the new product after \$500,000 is spent on advertising.

$\mathrm{f(x)} = 10{,}000 \cdot \log(x)$	Plug x = 1,000,000 into the function.
$\mathrm{f}(1{,}000{,}000) = 10{,}000 \cdot \log(1{,}000{,}000)$	Use a scientific calculator with a "log"
$\mathrm{f}(1{,}000{,}000) = 60{,}000$	button, which is log base 10.

About 60,000 people will have heard of the new product after \$1,000,000 is spent on advertising.

Since only 3000 more people will hear about the new product after spending another \$500,000, it is probably not worth spending the extra half million dollars on advertising.

10. For exponential growth, doubling time is the time that it takes for the amount to double. Doubling time is related to the continuous growth rate by $\ln 2 = \mathrm{k} \cdot \mathrm{t}$ where k is the growth rate and t is the doubling time. If it takes 30 years for prices to double, what is the growth rate, which is also known as inflation?

Steps	Reasons
$\ln 2 = \mathrm{k} \cdot \mathrm{t}$	Write the formula.
$\ln 2 = \mathrm{k} \cdot 30$	Replace t with the 30 year doubling time.
$\frac{\ln 2}{30} = \mathrm{k}$	Solve for k and use a scientific calculator to find k.

$\mathrm{k} = 0.0231\ldots$ or 2.3%

If the doubling time is 30 years, then inflation (or growth rate) is 2.3%.

Exercises

Write in logarithmic form:

1. $100 = 10^2$

2. $10{,}000 = 10^4$

3. $8 = 2^3$

4. $1 = 7^0$

5. $\frac{1}{9} = 3^{-2}$

6. $\frac{1}{25} = 5^{-2}$

Write in exponential form:

7. $2 = \log_4 16$

8. $3 = \log_5 125$

9. $0 = \log_5 1$

10. $-4 = \log_{10} 0.0001$

11. $-2 = \log_7 \frac{1}{49}$

12. $-4 = \log_3 \frac{1}{81}$

Graph the following:

13. $\mathrm{y} = \log_2 \mathrm{x}$

14. $\mathrm{y} = \log_4 \mathrm{x}$

15. $\mathrm{y} = \log_2(\mathrm{x} + 3)$

16. $\mathrm{y} = \log_2(\mathrm{x} - 1)$

17. $y = \log_{\frac{1}{3}} x$

18. $y = \log_{\frac{1}{4}} x$

19. $y = \log_3(x + 1)$

20. $y = \log_3(x - 2)$

Solve the following:

21. Advertising dollars are well spent up to a point. For instance a well-known company may decide to spend another ten million dollars, but it is unlikely that they will reach people that have not heard of their product. Suppose the function $f(x) = 1000 \cdot \log(x)$ models the number of people that will first hear of a particular new product for x equal to the amount spent on advertising in dollars. How many people will hear of the new product after $40,000 is spent on advertising? After $1,500,000? Is it worth spending the extra money on advertising?

22. Advertising dollars are well spent up to a point. For instance a well-known company may decide to spend another ten million dollars, but it is unlikely that they will reach people that have not heard of their product. The function $f(x) = 100{,}000 \cdot \log(x)$ models the number of people that will hear of a particular new product for x equal to the amount spent on advertising in dollars. How many people will have heard of the new product after $2,500,000 is spent on advertising? After $25,000,000? Is it worth spending the extra money on advertising?

23. For exponential growth, doubling time is the time that it takes for the amount to double. Doubling time is related to the continuous growth rate by $\ln 2 = k \cdot t$ where k is the growth rate and t is the doubling time. If it takes 7 years for prices to double, what is the growth rate, which is also known as inflation?

24. For exponential growth, doubling time is the time that it takes for the amount to double. Doubling time is related to the continuous growth rate by $\ln 2 = k \cdot t$ where k is the growth rate and t is the doubling time. If it takes 30 years for the population to double, what is the growth rate?

33. At most $9\frac{7}{9}$ ounces of artificial flavors can be added to the real orange juice.

35. $68° \le F \le 95°$

Exercise Set 4.1

1. $10x^3 + 15x^2 - 20x$

3. $21x^3 - 35x^2 - 77x$

5. $-6x^4 + 9x^3 + 18x^2$

7. $-8x^4 + 20x^3 + 24x^2$

9. $15x^8 - 9x^7 - 6x^6$

11. $12x^3y - 8x^2y^2 + 20xy^3$

13. $-45x^3y^3 + 30x^4y^2 + 24x^4y^3$

15. $40x^4y^2 - 80x^3y^2 - 56x^3y^3 - 96x^4y^3$

17. $15x^2 + 22x + 8$

19. $15x^2 + 29x - 14$

21. $35x^2 - 87x + 22$

23. $42x^2 - 53x + 15$

25. $55x^2 - 74x + 24$

27. $72x^2 - 15x - 42$

29. $15x^2 + 30x - 120$

31. $6x^4 - 6x^2 - 36$

33. $72x^4 - 206x^2 + 140$

Exercise Set 4.2

1. $6x(2x^2 + x - 3)$ or $6x(2x + 3)(x - 1)$

3. $4x^3(5x^2 - 4x - 3)$

5. $11x^2(3x^4 + 5x^2 - 4)$

7. $9x^2(7x^3 - 2x^2 + 6)$

9. $y^2(40y^4 - 49y^2 - 35)$

11. $(x + 2)(x + 3)$

13. $(x + 8)(x + 1)$

15. $(x - 4)(x + 2)$

17. does not factor

19. $(x - 3)(x - 4)$

21. $(x - 5)(x + 4)$

23. $(x + 7)(x - 3)$

25. $(x - 4)(x - 1)$

27. $(x + 9)(x + 2)$

29. $(x - 8)(x + 3)$

31. $(x + 6)(x - 5)$

33. does not factor

35. $(x + 8)(x + 8)$

37. $(2x - 1)(x + 3)$

39. $(3x - 1)(x - 5)$

41. does not factor

43. $(2x + 3)(x - 1)$

45. $(5x + 2)(x - 3)$

47. $(3x - 1)(2x + 1)$

49. $(4x+3)(x-2)$

51. $(2x-5)(2x+3)$

53. $5x(x+3)(x+1)$

55. $3x^2(x-5)(x+2)$

57. $4x^3(x-4)(x-2)$

59. $20x^3(x+4)(x+1)$

61. $x = 7,9$

63. $x = -\frac{3}{2}, \frac{1}{3}$

65. $x = 1,4$

67. $x = -7,3$

69. $x = -1,6$

71. $x = -3,4$

73. $x = 3,6$

75. $x = 0,1,2$

77. $x = -4,0,5$

79. $x = -\frac{1}{3}, 2$

81. $x = -\frac{3}{2}, 1$

83. The numbers are 5 and 6.

85. The numbers are 3 and 5.

Exercise Set 4.3

1. $x = -1, \frac{5}{3}$

3. $x = \frac{-3 \pm \sqrt{29}}{10}$

5. $x = \frac{7 \pm \sqrt{13}}{6}$

7. $x = \frac{3 \pm \sqrt{5}}{2}$

9. $x = \frac{1 \pm \sqrt{85}}{6}$

11. $x = \frac{1 \pm 2\sqrt{7}}{3}$

13. $x = -\frac{1}{6}, \frac{1}{2}$

15. $x = \frac{-1 \pm \sqrt{5}}{4}$

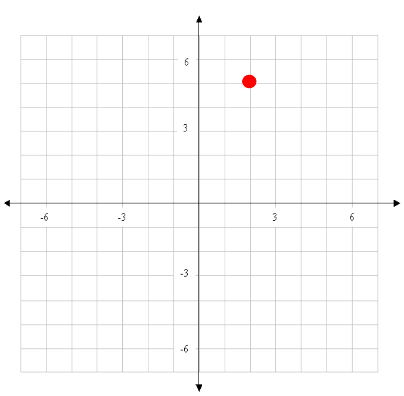
17. $x = -\frac{2}{3}, 2$

19. $x = \frac{7 \pm \sqrt{37}}{4}$

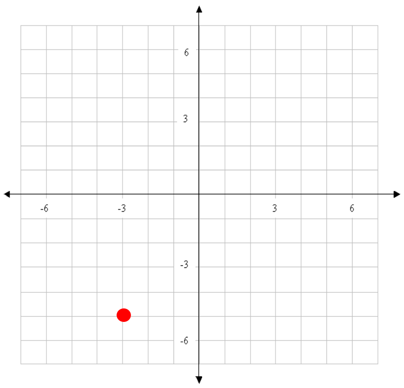
21. $x = 1 \pm \sqrt{6}$

Exercise Set 5.1

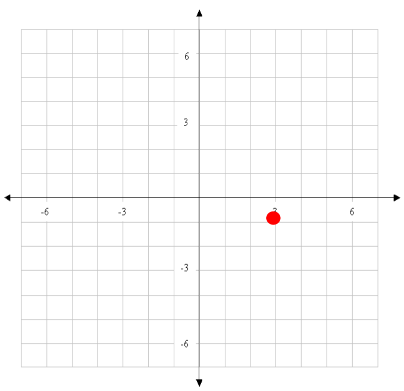
1. $(2,5)$



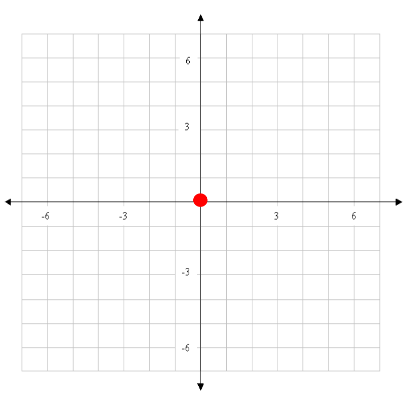
3. $(-3,-5)$



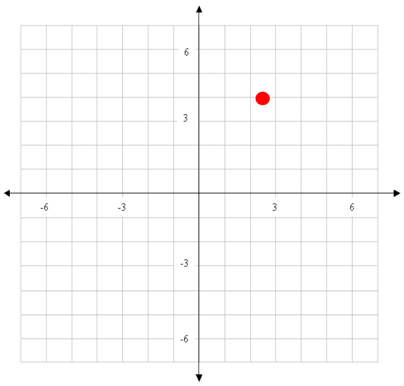
5. $(3,-1)$



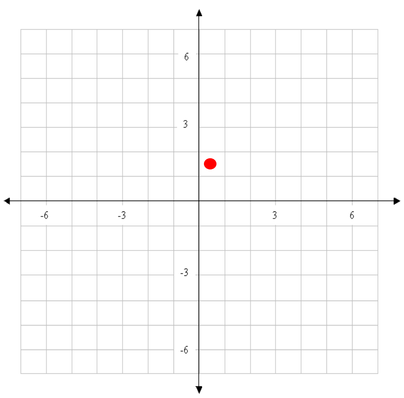
7. $(0,0)$



9. $(2.5, 3.5)$



11. $\left(\frac{1}{2}, \frac{3}{4}\right)$



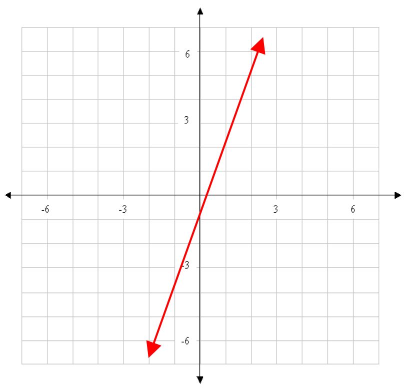
13. a. $f(2) = 1$
    b. $f(10) = 25$
    c. $f(-3) = -14$
    d. $f(0) = -5$

15. a. $g(2) = 14$
    b. $g(-3) = -6$
    c. $g(0) = 0$
    d. $g\left(\frac{1}{5}\right) = 1\frac{1}{25}$

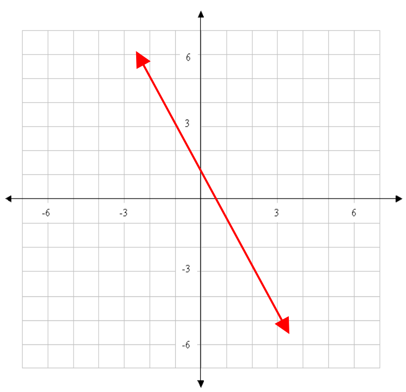
17. a. $f(3) = 35$
    b. $f(-2) = 20$
    c. $f(0) = 8$
    d. $f(-5) = 83$

19. a. $h(7) = 4$
b. $h(-5) = 2$
c. $h(0) = 3$
d. $h(91) = 10$

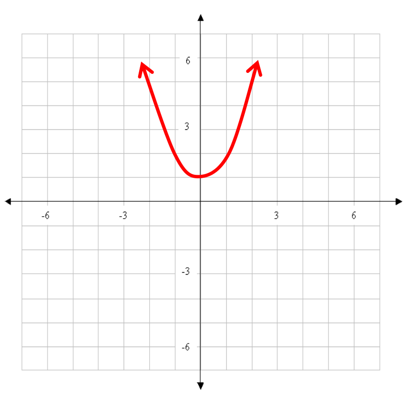
21. $f(x) = 3x - 1$



23. $f(x) = -2x + 1$



25. $g(x) = x^2 + 1$



27. Yes, the graph is the graph of a function.

29. Yes, the graph is the graph of a function.

31. No, the graph is not the graph of a function.

33. No, the graph is not the graph of a function.

35. Yes, the graph is the graph of a function.

37. a. 7725 kilograms
    b. 193,125 kilograms

39. a. $48,550
    b. The function underestimates the price by $3450.

Exercise Set 5.2

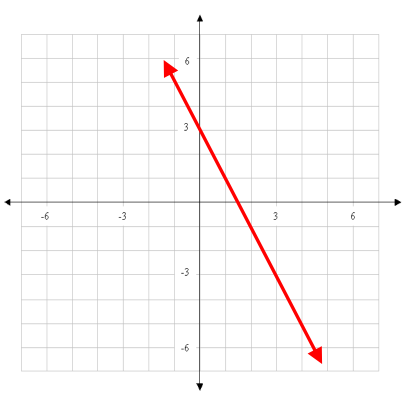
1. Yes, $(3,5)$ is a solution to $y = 2x - 1$.

3. No, $(5, -4)$ is not a solution to $y = -2x + 3$.

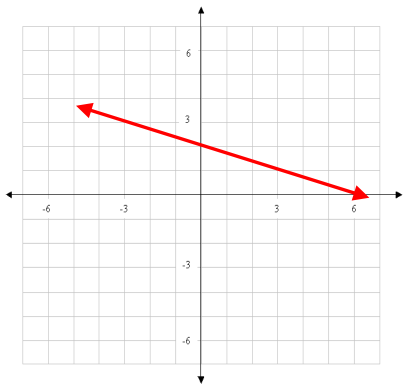
5. Yes, $(-2,3)$ is a solution to $y = 2x + 7$.

7. No, $(-2,5)$ is not a solution to $3x + 2y = 9$.

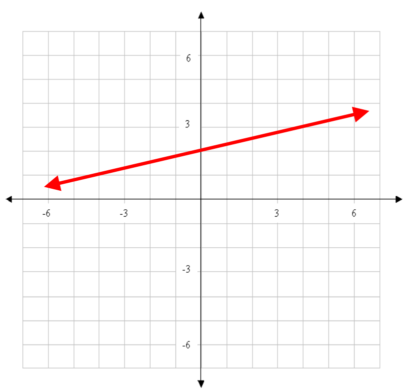
9. $y = -2x + 3$



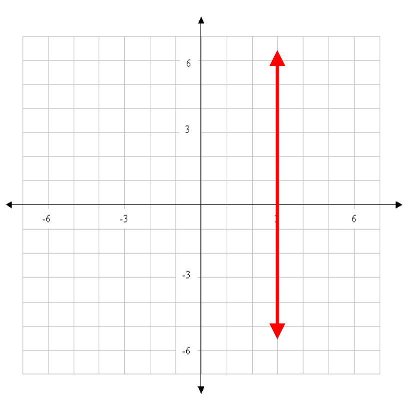
11. $y = -\frac{1}{3}x + 2$



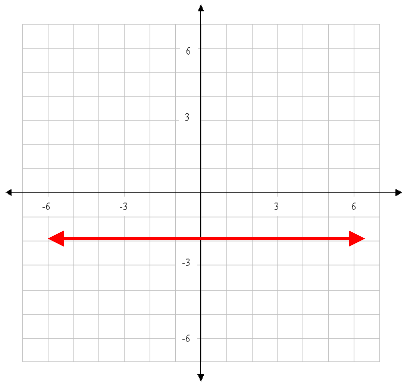
13. $y = \frac{1}{4}x + 2$



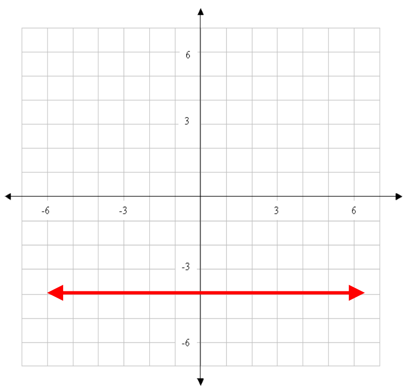
15. $x = 3$



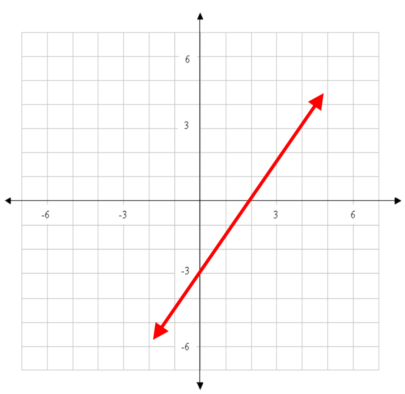
17. $y = -2$



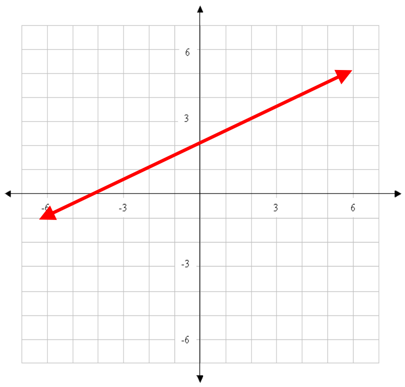
19. $y = -4$



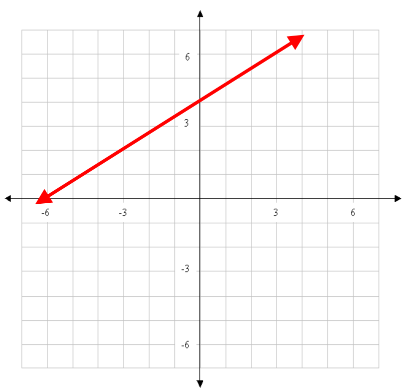
21. $3x - 2y = 6$



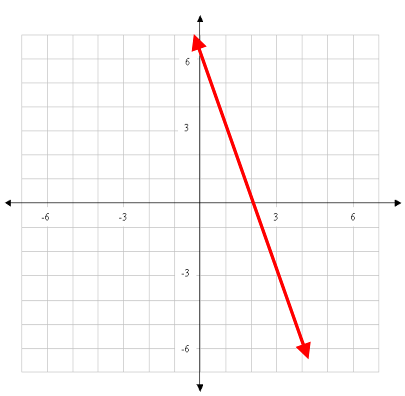
23. $2x - 4y = -8$



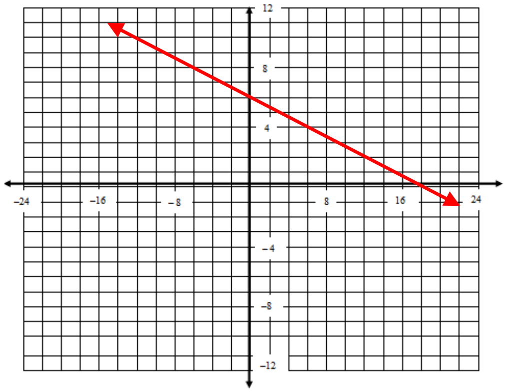
25. $-2x + 3y = 12$



27. $y = -3x + 6$



29. $y = -\frac{1}{3}x + 6$



31. $m = 4$

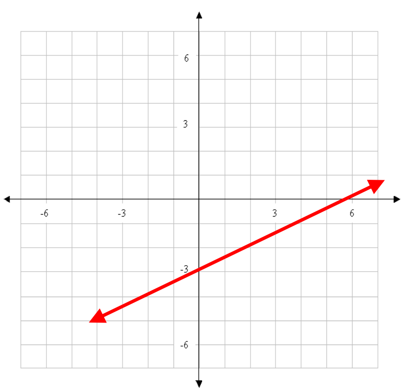
33. $m = -3$

35. $m = -\frac{1}{2}$

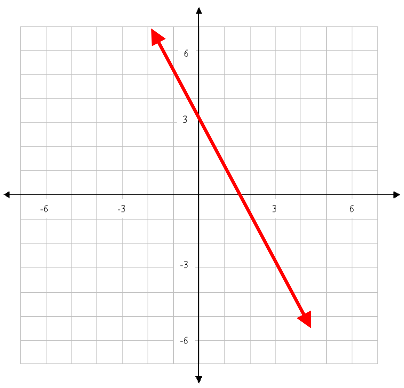
37. $m = 0$

39. undefined slope

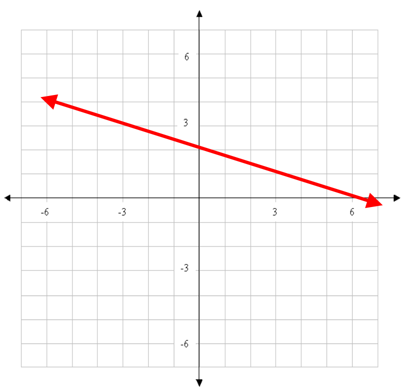
41. $y = \frac{1}{2}x - 3$



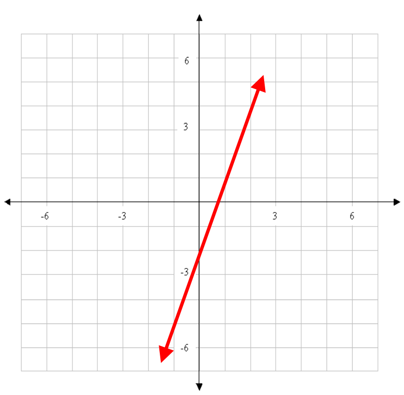
43. $y = -2x + 3$



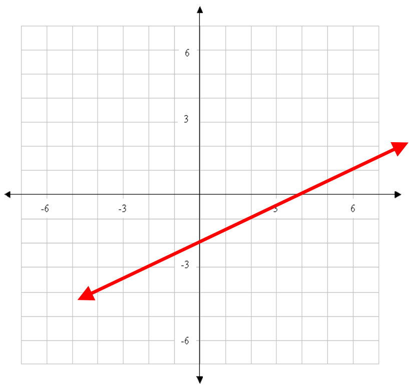
45. $y = -\frac{1}{3}x + 2$



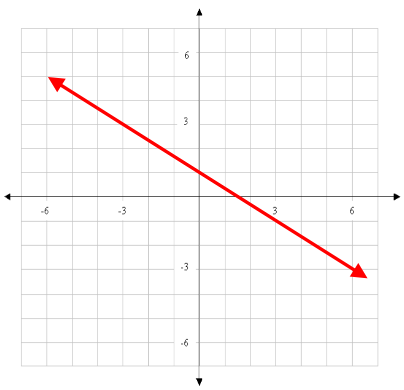
47. $y = 3x - 2$



49. $3x - 6y = 12$



51. $2x + 3y = 3$



Exercise Set 5.3

1. Yes

3. No

5. $(3,1)$

7. No solution

9. $(1,5)$

11. $(-1,1)$

13. $(2,3)$

15. $(-2,6)$

17. $(-5,-2)$

19. $(3,1)$

21. $(-3,2)$

23. $(1,4)$

25. $(2,-1)$

27. $(-1,-2)$

29. $(7,-3)$

31. $(-3,2)$

33. There are 52 grams of sugar in the cola and 45 grams of sugar in the root beer.

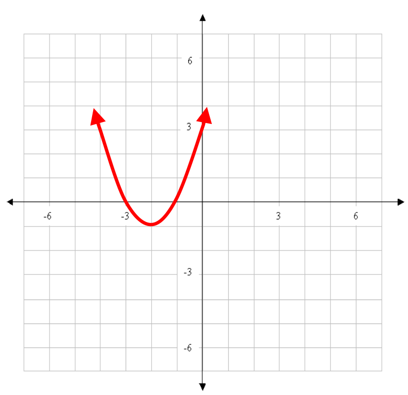
35. 21 motorcycles and 17 cars had all tires replaced.

37. The 4% bond is for $120,000 and the 2% bond is for $30,000.

39. The hot dog costs 1.75 euro and the beer costs 1.50 euro.

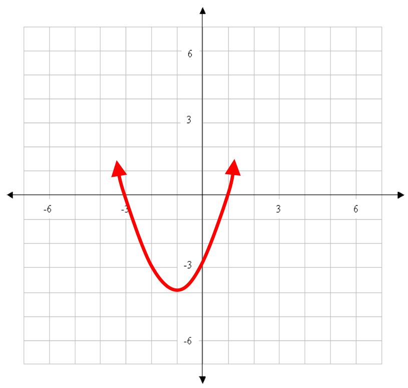
Exercise Set 5.4

1. $f(x) = x^2 + 4x + 3$
a. up
b. $(-2,-1)$
c. $(-3,0)$ and $(-1,0)$
d. $(0,3)$
e.



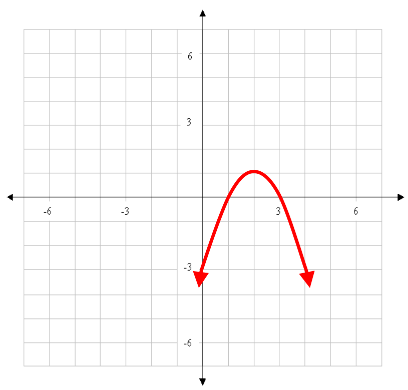
3. $f(x) = x^2 + 2x - 3$

a. up
b. $(-1, -4)$
c. $(-3,0)$ and $(1,0)$
d. $(0, -3)$
e.

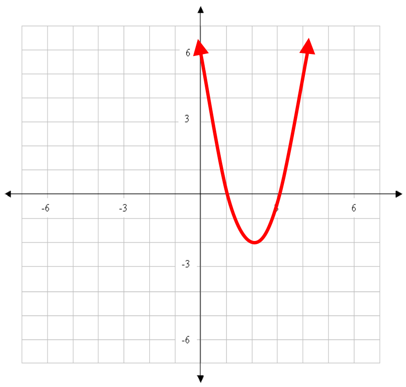


5. $f(x) = -x^2 + 4x - 3$

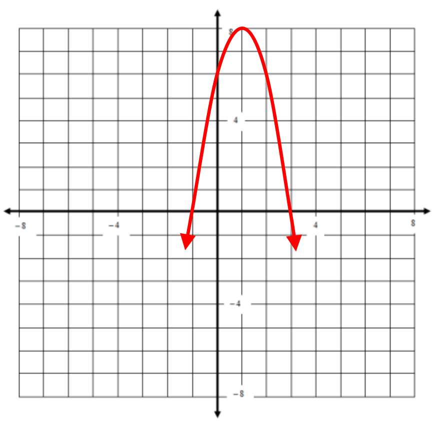
a. down
b. $(2,1)$
c. $(1,0)$ and $(3,0)$
d. $(0, -3)$
e.



7. $f(x) = 2x^2 - 8x + 6$
a. up
b. $(2, -2)$
c. $(1,0)$ and $(3,0)$
d. $(0,6)$
e.



9. $f(x) = -2x^2 + 4x + 6$
a. down
b. $(1,8)$
c. $(-1,0)$ and $(3,0)$
d. $(0,6)$
e.



11. The minimum of -34 is found at x = - 3.

13. The maximum of 42 is found at x = - 2.

15. The minimum of -140 is found at x = 4.

17. The maximum of 82,500 is found at x = 25.

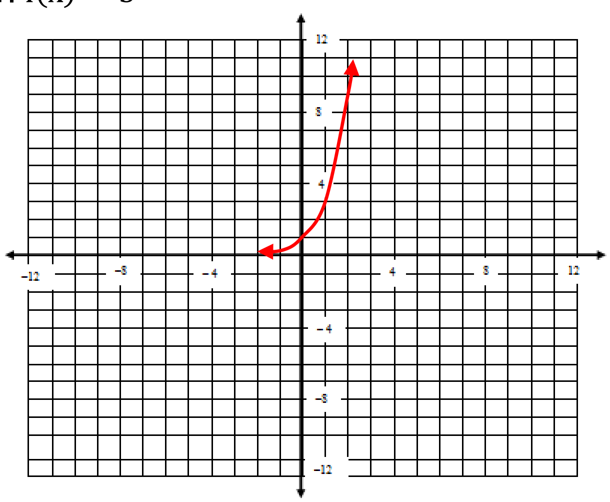
19. According to the function, the maximum profit is $15,988,000, which occurs when 20,000 security systems are installed.

21. According to the function, the maximum profit is $34,500, which occurs when 300 boots are produced and sold.

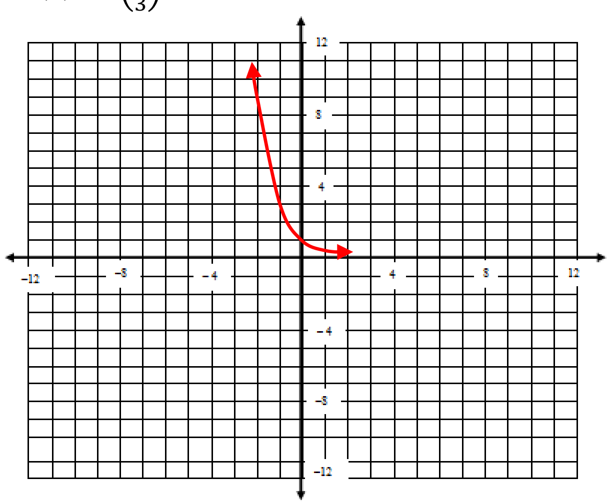
23. According to the function, the maximum height is 44.1 meters, which occurs 3 seconds after the object is launched. The time t must be more than 0 seconds because before that the object has not been launched. The time can be no more than 6 seconds because the object hits the ground 6 seconds after being launched.

Exercise Set 5.5

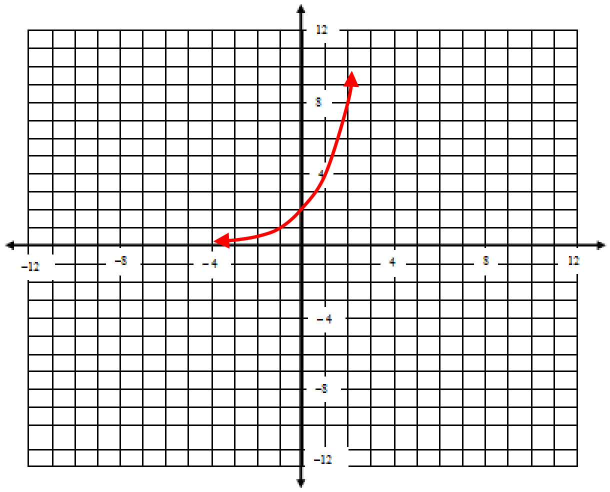
1. $f(x) = 3^x$



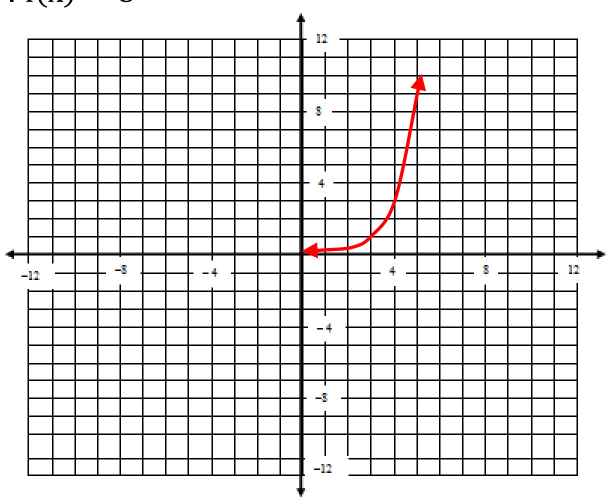
3. $f(x) = \left(\frac{1}{3}\right)^x$



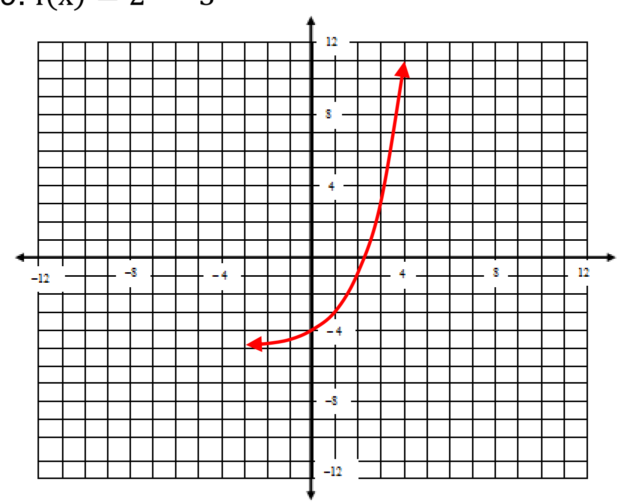
5. $f(x) = 2^{x+1}$



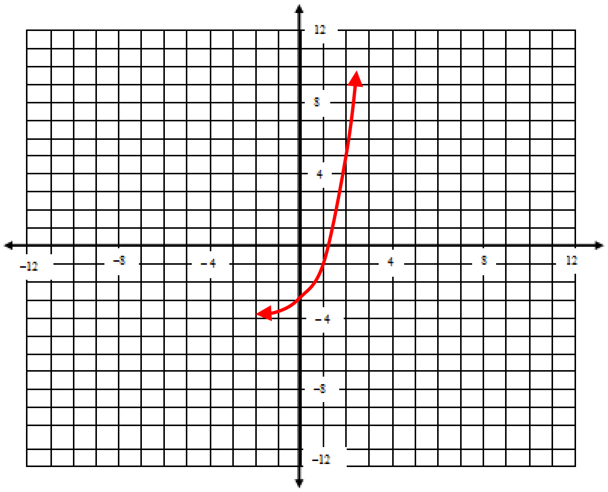
7. $f(x) = 3^{x-3}$



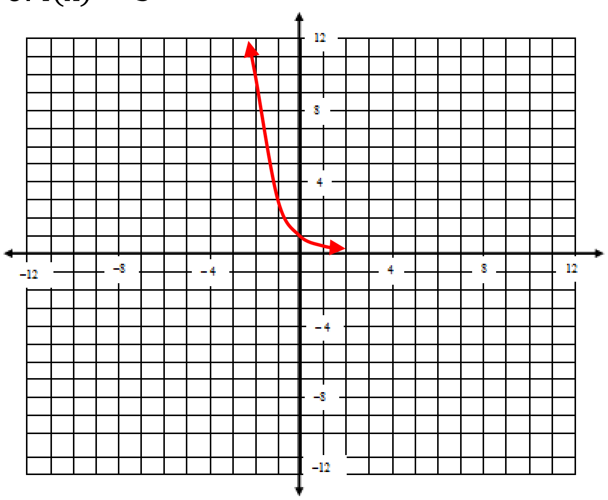
9. $f(x) = 2^{x} - 5$



11. $f(x) = 3^x - 4$



13. $f(x) = 3^{-x}$



15. According to the formula, 5,000,000 people had malaria in 1960 and 10,000,000 people had malaria in 1985. So, the doubling time is 25 years.

17. According to the formula, there will be 14.9 grams of carbon-14 10,000 years after the organism dies.

Exercise Set 5.6

1. $2 = \log_{10}100$

3. $3 = \log_2 8$

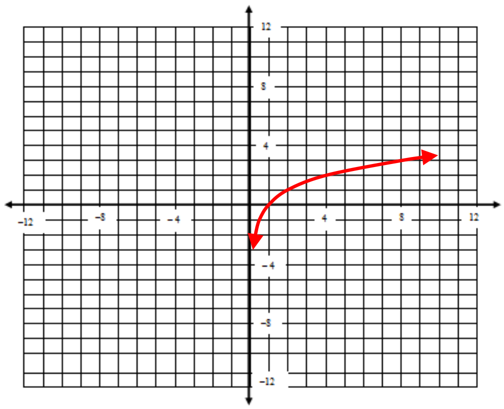
5. $-2 = \log_3 \frac{1}{9}$

7. $16 = 4^2$

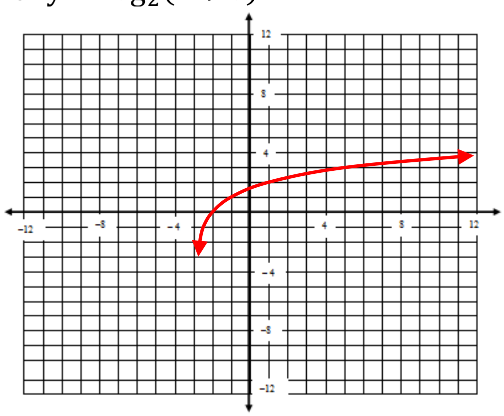
9. $1 = 5^0$

11. $\frac{1}{49} = 7^{-2}$

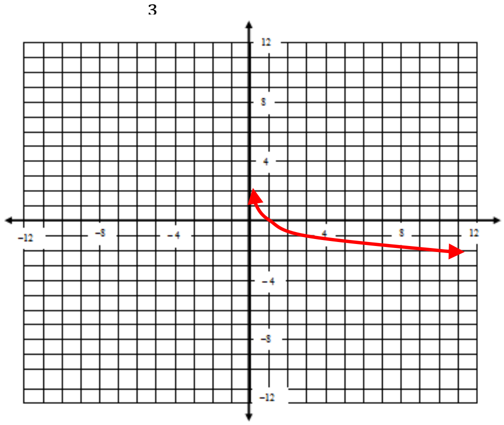
13. $y = \log_2 x$



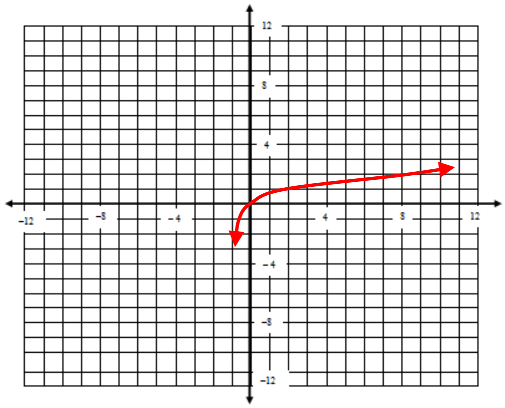
15. $y = \log_2(x + 3)$



17. $y = \log_{\frac{1}{3}} x$



19. $y = \log_3(x+1)$



21. 4602 people will hear about the product after $40,000 is spent on advertising. 6176 people will hear about the product after $1,500,000 is spent on advertising. It does not appear that the extra advertising dollars will be well spent.

23. The growth rate is 9.9%, which is very high for inflation.

Exercise Set 6.1

1. $2080

3. $225,000

5. $162

7. $580.83

9. $184.93

11. $16,695.21

13. $25,920

15. $15,750

17. $14,978

19. 6.5%

21. 7.6%

23. $12,791