Math For Today's World

ALGEBRA FINANCE STATISTICS

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Many students have difficulties in algebra and other types of mathematics because they lack proficiency in basic arithmetic skills. In Chapter 1, there is a review of these fundamental skills. **Mastery of Section 1.1 will be especially important in providing students with the requisite skills for being successful in the algebra portion of the course.**

Course Outcomes:

- Perform and execute basic arithmetic operations and simplify expressions involving exponents and square roots
- Recognize and apply mathematical concepts to real-world situations

1.1 Signed Numbers and Order of Operations

Students will learn about different types of numbers, absolute value, order of numbers, and how to perform basic arithmetic operations with integers. Exponents and various properties of real numbers are introduced. Mastering this section will give students important skills for the rest of the course.

1.2 Prime Factorization, GCF, LCM

Exponents, factors, prime factors, common factors, multiples, common multiples, and applications are covered. Students are introduced to these topics that are important when working with fractions and some applied problems.

1.3 Fractions

Rational numbers, proper fractions, improper fractions, mixed numbers, equivalent fractions, and simplest form are defined and used. Students learn to do arithmetic with fractions including with order of operations.

1.4 Square Roots

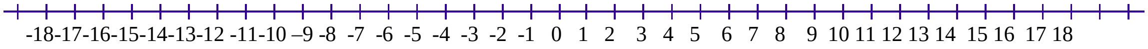
Square roots and irrational numbers are defined. Students learn to do arithmetic operations and applications with square roots.

1.5 Exponents and Scientific Notation

The definition of exponents is reviewed. The rules for exponents are presented and students learn to use them. Scientific Notation is presented and the rules of exponents are used to multiply and divide.

Perhaps the most typical notion of a number is what we do when we are counting. We start at 1 and continue: 1,2,3,4,5,.. These are called **natural numbers.** By adding zero to these natural numbers we get the **whole numbers,** which can be expressed as a set {0,1,2,3,4,5,…}. By now we are familiar with the arithmetic operations of addition, subtraction, multiplication, and division with whole numbers.

Integers include the whole numbers and their opposites or negatives. Fractions and decimals are not integers. When we start talking about arithmetic with integers (signed numbers both positive and negative) things get a little tricky. Even if students are comfortable with arithmetic and these signed numbers, they often make careless mistakes with negative signs. So, we will look at integers more closely. Integers can be listed on a number line as follows.



Remember: Numbers to the left on the number line are smaller than numbers to the right.

Statement	Symbolically
Negative four is less than six	$-4 < 6$
Negative two is greater than negative seven	$-2 > -7$
Three is greater than negative twelve	$3 > -12$

The absolute value of an integer is the distance to zero on the number line. Because distance is a positive value, we often just keep the integer positive (if it is positive) or make it positive (if it is negative). We use two vertical lines to indicate absolute value.

Examples

1. Evaluate $|-3|$. The two vertical lines indicate the absolute value of –3.

$$|-3| = 3$$

Make the number positive and that is it. Note that the –3 is three units from zero on the number line.

2. Evaluate $|5|$. The two vertical lines indicate the absolute value of 5.

$$|5| = 5$$

Keep the number positive and that is it. Note that the 5 is five units from zero on the number line.

3. Write $-3, |-7|, 2, -|-9|, -(-3), 12$ in order from smallest to largest.

Steps	Reason
$-3, \|-7\|, 2, -\|-9\|, -(-3), 12$	Simplify the numbers and write them below.
–3, 7, 2, –9, 3, 12	Write the simplified numbers in order from smallest to largest by looking at a number line if needed. Write the original form in that same order.
– 9, –3, 2, 3, 7, 12	
$-\|-9\|, -3, 2, -(-3), \|-7\|, 12$	

Rules for adding and subtracting integers:

Steps for adding two integers (Integers are positive or negative numbers with no fractional part):

Same sign (both numbers are positive or both are negative)
Size: add absolute values (add as if both numbers are positive)
Sign: same sign as addends (numbers being added)

Different Signs (one negative another positive number are added)
Size: subtract the absolute values
Sign: same sign as the number with larger absolute value

We should ask ourselves where these rules come from? Do they make sense? Adding two positive numbers is familiar. When we add two negative numbers we can take the example of adding two debts. If somebody owes one friend $5 and another friend $6, then the person owes $11. So, - 5 + (-6) = -11.

For adding integers with different signs, we can think of money as well. Somebody that has $5 but owes $3 really has $2, which is to say 5 + (-3) = 2. However, if somebody has $5 and owes $8 then they have a negative $3. This would be the same as 5 – 8 = -3 or 5 + (-8) = -3. Once we accept the rules we need to apply them with confidence.

Examples

4. Add: –15 + 6

Steps	Reason
–15 + 6	Adding one positive and one negative integer
–9	Size: Subtract the absolute values. 15 – 6 = 9 Sign: Because 15 is larger the sign is negative.

5. Add: –9 + (–14)

Steps	Reason
–9 + (–14)	Adding two negative integers. (Same sign)
–23	Size: Add the absolute values. 9 + 14 = 23 Sign: Negative for adding two negative numbers.

Steps for **subtracting two integers:**

Subtraction can be rewritten as adding the opposite of a number. So,

a – b = a + (–b)

Thinking of subtraction as adding the opposite lets us use the rules for adding two integers.

Examples

6. Subtract: 7 – 15

Steps	Reason
7 – 15	If we cannot subtract in one step, rewriting the problem lets us discuss the steps.
7 + (–15)	Rewrite subtracting 15 as adding –15
–8	Size: Take the difference of (subtract) the absolute values. Sign: Since 15 is larger, the sign is negative.

7. Subtract: 18 – 11

Steps	Reason
18 – 11 = 7	Just do the subtraction. It would confuse things to rewrite this problem.

8. Subtract: –7– 11

Steps	Reason
–7– 11	If you cannot do it in one step, rewriting the problem lets us discuss the steps.
–7+ (–11)	Write subtracting 11 as adding –11.
–18	Size: Take the sum of (add) the absolute values. 7 + 11 = 18 Sign: Since both are negative, sign is negative.

The above three examples can be done in one step. By thinking of the subtraction as adding the opposite, we can apply the rules for adding signed numbers.

When we combine subtraction with negative numbers sometimes the symbols look awkward. Never fear, we can simplify:

Statement	Symbols	Simplified
The opposite of negative eight	– (–8)	8
Twelve minus negative five	12– (–5)	12 + 5
9 plus negative seven	9 + (–7)	9 – 7

Making the opposite of a negative number positive and changing minus a negative number into addition can really help to simplify some otherwise complex expressions.

Example

9. Simplify 11 – (–2) – 18 + 4

Steps	Reason
11 – (–2) – 18 + 4	Write the original problem.
11 + 2 – 18 + 4	Take a step to change “minus a negative” to “plus”.
13 – 18 + 4	Add and subtract from left to right.
–5 + 4	13 – 18 is the same as 13 + (–18). Take the difference 18 – 13 = 5 and the sign is negative.
–1	For –5 + 4, take the difference 5 – 4 and the sign is negative.

Rules for multiplying and dividing integers:

Steps for **multiplying two integers:**

Size: multiply the absolute values (ignore negative signs) of the factors

Sign:

- Both positive factors yield positive product: 2 · 3 = 6
- Both negative factors yield a positive product: (–4)(–6) = 24
- One negative factor and one positive factor yield a negative product: (–3)·5 = –15

Steps for **dividing two integers:**

Same steps as multiplication except do division.

36 ÷ (–4) = –9	Divide as if the numbers are positive. One number negative and another positive yields a negative quotient. (Quotient is the result of division.)
(–40)÷(–8) = 5	Divide as if the numbers are positive. Because both numbers are negative, the quotient is positive.

<u>Examples</u>

10. Evaluate: $-72 \div (-9)$

Steps	Reason
$-72 \div (-9) = 8$	72 ÷9 is 8. Because both -72 and -9 are negative, the quotient is positive.

11. Evaluate –(–3)(–2)(–7).

Steps	Reason
–(–3)(–2)(–7)	Multiply the numbers as if they were positive. Because there are four negative signs the product is positive.
42	

12. Find the quotient of –92 and 4.

Steps	Reason
–92 ÷ 4	Rewrite quotient as division.
–23	92 ÷ 4 is 23. Because one of the numbers is negative and the other is positive, the quotient is negative.

<u>The Multiplication Property of Zero:</u>

A · 0 = 0 or 0 · A = 0

Multiply any number by zero and get zero. For example, 3 · 0 = 0.

The Multiplication Property of One:

$A \cdot 1 = A$ or $1 \cdot A = A$

Multiply any number by one and get the same number. $1 \cdot 7 = 7$ We can call 1 the identity of multiplication because it gives us what we started with when we multiply.

The Commutative Property of Multiplication:

$A \cdot B = B \cdot A$

We can multiply in either order. Both $5 \cdot 3$ and $3 \cdot 5$ are 15

The Associative Property of Multiplication:

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$

If there is only multiplication, the numbers can be grouped either way.

$(4 \cdot 2) \cdot 5$	$4 \cdot (2 \cdot 5)$
$= 8 \cdot 5$	$= 4 \cdot 10$
$= 40$	$= 40$

Both ways we get 40.

Using the Commutative Property and the Associative property, any list of integers that are multiplies can be multiplied in any order.

Division properties

$\frac{0}{a} = 0$, Zero divided by any number is zero. $\frac{0}{12} = 0$

$\frac{a}{a} = 1$, Any number divided by itself is one. $\frac{9}{9} = 1$

$\frac{a}{1} = a$, Any number divided by one is itself. $\frac{6}{1} = 6$

$\frac{a}{0}$ is undefined. Never divide by zero. It is impossible. $\frac{5}{0}$ is undefined.

The Addition Property of Zero:

A + 0 = A or 0 + A = A
In other words, if you add zero you get what you started with. 5 + 0 = 5

We can call zero the identity for addition because we get what we started with when we add 0.

The Commutative Property of Addition:

A + B = B + A

We can add in either order. Both 5 + 3 and 3 + 5 are 8

The Associative Property of Addition:

(A + B) + C = A + (B + C)

If there is only addition the numbers can be grouped either way.

	(4 + 2) + 5		4 + (2 + 5)
=	6 + 5	=	4 + 7
=	11	=	11

Both ways we get 11.

The Inverse Property of Addition

A + (–A) = 0 or –A + A = 0

If a number is added to its opposite, the sum is 0. For example –5 + 5 = 0.

Using the Commutative Property and the Associative property, any list of integers to be added can be added in any order. For example,

$\left(-2\right)+9+12+1=9+1+12+\left(-2\right)$ We can change the order by the commutative property to add in any order that we like, which makes it easier to add here.

Being able to add or multiply in any order will be helpful in algebra. So remember:
If there is a list of numbers that is added together we can add in any order.
If there is a list of numbers multiplied together we can multiply in any order.

Exponents can be used to write repeated multiplication. $5^3 = 5 \cdot 5 \cdot 5$ The exponent of 3 tells us how many times to multiply the base, which is 5.

Examples

13. Find 2^4

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

14. Find $(-3)^2$

$$(-3)^2 = (-3)(-3) = 9$$

15. Find -4^2

$-4^2 = -4 \cdot 4 = -16$ This case catches many people. The trick is that the exponent only goes with the symbol in front. So, here the exponent only goes with the 4 and the negative sign means that 4^2 will be negative. If we want a negative number to be squared, then we need parentheses like the example with $(-3)^2$.

The **order of operations** is fundamental to most problems in algebra. We need to know them and be able to use them comfortably. The **order of operations** is as follows:

1. **Parentheses** inside to outside.
2. **Exponents.**
3. **Multiplication** and **division** together as they appear from left to right.
4. **Addition** and **subtraction** together as they appear from left to right.

Some people remember **PEMDAS** to help keep the order straight. Back in the day you may have heard Please Excuse My Dear Aunt Sally as a way to remember PEMDAS. Eventually, we want to be completely comfortable with the above order of operations. Be careful! Division and multiplication are at the same level. If the division comes before multiplication as we read left to right, then we do that division first. Addition and subtraction are similar in this. If subtraction is to the left of addition with no other symbols or operations, we do the subtraction before the addition.

Examples

16. Simplify: $4 + 3(5^2 – 15)$

Steps	Reasons
$4 + 3(5^2 – 15)$	Think about the operations that appear and their order.
$4 + 3(25 – 15)$	
	1. Work within parentheses. Evaluate the exponent before the subtraction.
$4 + 3(10)$	
	2. Keep working within the parentheses by doing the subtraction.
$4 + 30$	
	3. Do multiplication before addition.
34	4. Add.

17. Simplify: $(-6)^2 \div 4 + 5 \cdot (-2)$

Steps	Reasons
$(-6)^2 \div 4 + 5 \cdot (-2)$	Look at the operations: exponents, division, addition, and multiplication. Do exponents.
$36 \div 4 + 5 \cdot (-2)$	
	Do division and multiplication at the same time.
$9 + (-10)$	
	Adding signed numbers
-1	

To check this problem you need to redo it on a separate piece of paper.

18. Simplify $-12(6-8)+1^3 \cdot 3^2 \cdot 2 - 2(-12)$

Steps	Reasons
$-12(6-8)+1^3 \cdot 3^2 \cdot 2 - 2(-12)$	Look at the operations: multiplication, subtraction, addition, and exponents. Work inside parentheses.
$-12(-2)+1^3 \cdot 3^2 \cdot 2 - 2(-12)$	
$-12(-2)+1 \cdot 9 \cdot 2 - 2(-12)$	Do exponents: $1^3 = 1$ and $3^2 = 3 \cdot 3 = 9$
$24 + 18 + 24$	Multiply: $-12(-2) = +24$ $1 \cdot 9 \cdot 2 = 18$ $-2(-12) = +24$
66	Add

To check this problem you need to redo it on a separate piece of paper.

Note: In each step one operation is being performed and everything else is being rewritten. Always rewrite everything you intend to simplify in later steps. Some operations can be performed slightly out of order. The parentheses and the exponents could have been done together in the last problem because addition separates them, but be careful.

19. Evaluate: $7-3\left[1-\left(2-(-3)\right)^2\right]$

Steps	Reasons
$7-3\left[1-\left(2-(-3)\right)^2\right]$	Write minus a negative as plus first because it is inside the parentheses.
$7-3\left[1-(2+3)^2\right]$	Then add, 2+3=5
$7-3\left[1-5^2\right]$	Square the 5 before subtraction.
$7-3[1-25]$	Subtract in the brackets.
$7-3[-24]$	Multiply before subtracting. -3 times -24 is +72
$7+72$	Add to get the answer
79	

Signed numbers and the order of operations are very important throughout all of mathematics. The order of operations gives the math world a structure to follow and signed numbers always need attention since many careless errors involve negative signs.

Exercises:

Place the correct symbol > (greater than), < (less than), or = (equal to) in the □ between the two numbers:

1. $7 \square 9$

2. $5 \square -4$

3. $-8 \square -12$

4. $-9 \square 0$

5. $-4 \square -3$

6. $|-3| \square 3$

7. $12 \square |-15|$

8. $|-5| \square 11$

9. $|-6| \square |4|$

Order from smallest to largest:

10. $|-8|, -4, 0, 6, |3|, |-10|$

11. $7, |-11|, -1, 3, |-2|, |0|$

12. $|-3|, -9, 6, |12|, 0, |-14|$

13. $-2, 1, |-6|, -5, 0, |11|, |-7|$

Simplify without using a calculator:

14. $-3 + 8$

15. $5 + (-7)$

16. $-12 + 5$

17. $-15 + (-6)$

18. $-8 + (-4)$

19. $18 + (-14)$

20. $-19 + (-16)$

21. $-25 + (-17)$

22. $-12 - 5$

23. $15 - 22$

24. $-3 - (-11)$

25. $-20 - (-11)$

26. $-10 - 12$

27. $12 - 24$

28. $-8 - (-17)$

29. $-14 - (-9)$

30. $-6 \cdot (-9)$

31. $5 \cdot (-9)$

32. $-8 \cdot (-7)$

33. $-4 \cdot 10$

34. $9 \cdot (-3)$

35. $-7 \cdot 9$

36. $-9 \cdot (-8)$

37. $81 \div (-9)$

38. $35 \div (-7)$

39. $-72 \div 8$

40. $-64 \div (-8)$

41. $-49 \div 7$

42. $-63 \div (-7)$

43. $100 \div (-5)$

44. $-36 \div (-4)$

45. 5^3

46. 6^2

47. $(-4)^2$

48. -2^3

49. $(-8)^2$

50. -3^2

51. -2^4

52. $(-7)^2$

53. $2 + 5(3)$

54. $15 - 3 \cdot 4$

55. $3 + 18 \div (-3)$

56. $-5 - (-12) \div (-4)$

57. $4 - (-21) \div (-3)$

58. $7 + 2(10 - 8)$

59. $-3 - 5(3 + 2)$

60. $8 - 5(3 - 7)$

61. $-10 - 4(3 - 7)$

62. $(5 - 2^2) \cdot (3^2 - 6)$

63. $(3 - 4^2) - (2^3 - 10)$

64. $8 - 4[3(5 - 2) - (2 - 4)]$

65. $3 + 2[4(2 - 3^2) - 2(5 - 4^2)]$

66. $6 + 5[2(5 - 2^3) + 3(-20 + 5^2)]$

67. $12 - 3[3(8 - 2^3) - 2(15 - 3^2)]$

68. $3(5-2^2) - 4(3^2 - 12)$

69. $3\{5 - 2[-5 - (-8)] - 3[-2^2 + 4(5 - 2)]\}$

70. $-2\{3[5 + (-7)] - 4[-2^4 + 2(-3)^2]\}$

71. $-4\{2[-3 + (-7)] - 4[2^3 + 4(-2)^3]\}$

For 72 and 73, remember that when we borrow or spend money we can use negative numbers and when we receive money we can use positive numbers.

72. Jack starts the weekend with $20. He then borrows $7 from each of six different friends and spends the money immediately. Another friend pays Jack back $15. Jack also spends $40 over the weekend. How much money does Jack have at the end of the weekend?

73. Maria started the month with $800. Each of the four weeks she spent $230 on household expenses. For the month she spent $275 on entertainment and eating out. How much did she have to borrow from her roommate to make it to the end of the month?

74. A diver goes down 20 feet in the first 5 minutes, 25 feet the second five minutes, and then 11 feet the next five minutes to reach the bottom. When the diver comes up he moves 8 feet every 5 minutes and he stops for 3 minutes every 8 feet. How long does it take the diver to go down? How long does it take the diver to return to the surface?

75. Rome was founded in 753 BC. The United States declared independence in 1776 AD. Use the appropriate negative number with subtraction to find the difference in time between the founding of Rome and the Declaration of Independence. Hint: Do not forget to take into account that there is no year 0 since we went from 1 BC to 1 AD.

Exponents can be used to write repeated multiplication. $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$ The exponent of 4 tells us how many times to multiply the base, which is 3.

Factors are multiplied to get the **product**. Since 3·5=15, 3 and 5 are factors of 15. Also, 1 and 15 are factors of 15 because 1·15=15. In the end 1,3,5,15 are all factors of 15.

Prime numbers have only two whole number factors which are 1 and the number itself. Examples of prime numbers are 2,3,5,7,11,13,17,19,23,29,31,…

We can write the **prime factorization** of a number by dividing the number by prime factors repeatedly.

<u>Example</u>

1. Find the prime factorization of 150.

$$\begin{array}{r} 75 \\ 2\overline{)150} \end{array}$$

Begin by dividing by the smallest prime number 2.

$$\begin{array}{r} 25 \\ 3\overline{)75} \\ 2\overline{)150} \end{array}$$

Divide by the next prime number 3.

$$\begin{array}{r} 5 \\ 5\overline{)25} \\ 3\overline{)75} \\ 2\overline{)150} \end{array}$$

Divide by the next prime number 5.

When the last factor is prime, we stop. The prime factorization is the product of the factors (or divisors) 2,3,5, and the remaining prime factor 5.

Since the 5 is a factor twice, the prime factorization is $2 \cdot 3 \cdot 5^2$.

Note: 2,3, and 5 are all prime numbers and the product $2 \cdot 3 \cdot 5^2 = 2 \cdot 3 \cdot 5 \cdot 5$ does equal 150.

The **multiples** of a number have the number as a factor.

Multiples of 6 are: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, …
Multiples of 9 are: 9, 18, 27, 36, 45, 54, 63, 72, 81,…

Common Multiples are multiples of two or more numbers.

The common multiples of 6 and 9 are 18, 36, 54, 72, …

The **Least Common Multiple (LCM)** is the smallest common multiple. The least common multiple of 6 and 9 is 18.
Note: The least common multiple of 6 and 9 is always at least as big as the larger of the numbers. 18 is a multiple of 6 because 6 · 3 = 18, and 18 is a multiple of 9 because 9 · 2 = 18.

To find the Least Common Multiple we can look at the multiples of the larger number until we find a multiple of the smaller number. We should also be able to find the LCM by first taking the prime factorization.

Finding the LCM by using the prime factorization:
1. First find the prime factorization as above.
2. Write all prime factors once and ignore the exponent.
3. Go back and chose the **largest** exponent for each of the prime factors.

Examples

2. Find the least common multiple of 36 and 30 by first finding the prime factorizations.

Steps	Reasons
$\begin{array}{r} 3 \\ 3\overline{)9} \\ 2\overline{)18} \\ 2\overline{)36} \end{array}$	Prime factorization of 36: Divide by 36 by 2 because 2 is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$36 = 2^2 \cdot 3^2$	The divisors and remainder make the prime factorization of 36.
$\begin{array}{r} 5 \\ 3\overline{)15} \\ 2\overline{)30} \end{array}$	Prime factorization of 30: Divide 30 by 2 because 2 is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$30 = 2 \cdot 3 \cdot 5$	The divisors and remainder make the prime factorization of 30.
$2 \cdot 3 \cdot 5$	Write all prime factors once and ignore the exponent.
$2^2 \cdot 3^2 \cdot 5$	The **largest** exponents for the 2 and 3 are two from the prime factorization of 36
180	Multiply $2^2 \cdot 3^2 \cdot 5$. So, 180 is the LCM of 36 and 30.

3. Find the least common multiple of 8, 12, and 27 by first finding the prime factorizations.

Steps	Reasons
$\begin{array}{r} 2 \\ 2\overline{)4} \\ 2\overline{)8} \end{array}$	Prime factorization of 8: Divide 8 by 2 because it is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$8 = 2^3$	The divisors and remainder make the prime factorization of 8.
$\begin{array}{r} 3 \\ 2\overline{)6} \\ 2\overline{)12} \end{array}$	Prime factorization of 12: Divide 12 by 2 because it is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$12 = 2^2 \times 3$	The divisors and remainder make the prime factorization of 12.
$\begin{array}{r} 3 \\ 3\overline{)9} \\ 3\overline{)27} \end{array}$	Prime factorization of 27: Divide 27 by 3 because it is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$27 = 3^3$	That is the prime factorization of 27.
$2 \cdot 3$	Write all prime factors once and ignore the exponent.
$2^3 \cdot 3^3$	The **largest** exponent for the 2 is the three from $2^3 = 8$ and the **largest** exponent for the 3 is the three from $3^3 = 27$.
216	Multiply $2^3 \cdot 3^3$. So, 216 is the LCM of 8, 12 , and 27.

Factors of a number divide the number evenly.

The factors of 12 are 1, 2, 3, 4, 6, and 12.
The factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30

The **common factors** of 12 and 30 are 1, 2, 3, and 6.

The **Greatest Common Factor (GCF)** of 12 and 30 is 6.

Careful, students tend to confuse Least Common Multiple and Greatest Common Factor. The Greatest Common Factor is always smaller than the numbers because it is a common factor for the numbers. The Least Common Multiple is always larger than the numbers because it is a common multiple of the numbers.

Finding the GCF by using the prime factorization:

1. First find the prime factorization as above
2. Write all the **common** prime factors once and ignore the exponent.
3. Go back and chose the **smallest** exponent for each of the prime factors.

The only change is that now we use the smallest exponent for each factor.

Example:

4. Find the greatest common factor of 24, 48, and 72 by first finding the prime factorizations.

Steps	Reasons
3 $2\overline{)6}$ $2\overline{)12}$ $2\overline{)24}$	Prime factorization of 24: Divide 24 by 2 because 2 is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$24 = 2^3 \cdot 3$	The divisors and remainder make the prime factorization of 24.
3 $2\overline{)6}$ $2\overline{)12}$ $2\overline{)24}$ $2\overline{)48}$	Prime factorization of 48: Divide 48 by 2 because it is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$48 = 2^4 \cdot 3$	That is the prime factorization of 48.
3 $3\overline{)9}$ $2\overline{)18}$ $2\overline{)36}$ $2\overline{)72}$	Prime factorization of 72: Divide 72 by 2 because it is the smallest prime factor. Keep dividing by the smallest prime factor until the remainder is prime.
$72 = 2^3 \cdot 3^2$	That is the prime factorization of 72.
$2 \cdot 3$	Write all common prime factors once and ignore the exponent.
$2^3 \cdot 3$	The **smallest** exponent for the 2 is the three from the 24 and the 72. The **smallest** exponent for the 3 is one from the 24 and 48.
24	Multiply $2^3 \cdot 3$. So, 24 is the GCF of 24, 48, and 72.

Note: 24, 48, and 72 all have 24 as a factor.

Exercises:

1. Find the prime factorization of 36.

2. Find the prime factorization of 45.

3. Find the prime factorization of 100.

4. Find the prime factorization of 360.

5. Find the prime factorization of 220.

6. Find the prime factorization of 720.

7. Find the least common multiple (LCM) of 18 and 45.

8. Find the least common multiple (LCM) of 42 and 70.

9. Find the least common multiple (LCM) of 120 and 216.

10. Find the least common multiple (LCM) of 24, 42, and 60.

11. Find the least common multiple (LCM) of 84, 108, and 120,

12. Find the least common multiple (LCM) of 504, 756, and 924.

13. Find the greatest common factor (GCF) of 18 and 24.

14. Find the greatest common factor (GCF) of 45 and 63.

15. Find the greatest common factor (GCF) of 168 and 280.

16. Find the greatest common factor (GCF) of 36, 90, and 108.

17. Find the greatest common factor (GCF) of 84, 168, and 252.

18. Find the greatest common factor (GCF) of 162, 270, and 486.

19. For a school lunch program, fruits can be bought in lots of 42, small boxes of milk can be bought in boxes of 30, and sandwiches can be bought in packages of 18. If each student receives one fruit, one box of milk, and one sandwich, what is the least number of students that can be served so that there are no leftover fruits, milk, or sandwiches?

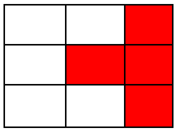
20. Alonso can drive around a race track in 6 minutes and Hamilton takes 8 minutes to drive around the same track. If they both start at the same time, how long will it take until Alonso and Hamilton pass the starting point together?

21. Jeff has 72 exams and Jack has 48 exams to grade. Both Jeff and Jack want to put their exams in equal sized groups so that the teaching assistants can grade the exams for them. What is the largest number of exams that they can put in each group so that each group is the same size?

22. Miss Olga has 90 lollipops, 126 candy canes, and 108 chocolate bars that she wants to give away in her classes. If she wants to split each of the candies into same sized piles with none leftover, what is the largest number of lollipops, candy canes, and chocolate bars that she can have in each pile?

Rational numbers are numbers that can be written as a quotient (or fraction) of two integers. Remember, **integers** are whole numbers and their opposites or negatives. Examples of rational numbers include $\frac{1}{2}, \frac{7}{3}, 5, -11, 4\frac{1}{8}, 0, etc.$ The integers are all rational numbers because they can be written as a fraction of integers. For instance, $5=\frac{5}{1}$ and $-11=\frac{-11}{1}$.

Fractions are used to represent equal parts of a whole.



There are 4 out of 9 of the equal-sized boxes are shaded. We can represent the shaded part of the whole figure as the fraction $\frac{4}{9}$. Here, the **numerator** is 4, and the **denominator** is 9.

Proper fractions have a value less than one.

Examples of proper fractions are $\frac{2}{3},\frac{3}{4},\frac{7}{24}$. The numerator is less than the denominator.

Improper fractions have a value greater than or equal to one.

Examples of improper fractions are $\frac{5}{3},\frac{7}{7},\frac{19}{3}$. The numerator is greater than or equal to the denominator.

Mixed numbers have a whole number part and a proper fraction part.

Examples of mixed numbers are $3\frac{1}{2}, 5\frac{3}{4}, 11\frac{7}{23}$. The value of the mixed number is the sum of the whole number and the proper fraction. So, $3\frac{1}{2}=3+\frac{1}{2}$, $5\frac{3}{4}=5+\frac{3}{4}$, and $11\frac{7}{23}=11+\frac{7}{23}$

Writing a mixed number as a improper fraction:

1.New Numerator:

Multiply whole number by the denominator and add the numerator.

2. New denominator:

Keep old denominator.

Example

1. Write $5\frac{2}{3}$ as an improper fraction.

Steps	Reasons
$5\cdot 3+2=17$	Multiply whole number by the denominator and add the numerator.
$\dfrac{17}{3}$	Keep old denominator.

This process works because $5\frac{2}{3}$ means $5+\frac{2}{3}$. To add the fractions we need a common denominator, which we do as follows $\dfrac{5}{1}\cdot\dfrac{3}{3}+\dfrac{2}{3}$. Looking at how fractions are added shows us why we multiply $5\cdot 3+2$ to get the new numerator and keep the old denominator when changing the mixed number to an improper fraction.

Write an improper fraction as a mixed number

Do long division.

The mixed number is $quotient\,\frac{remainder}{divisor}$.

Example

2. Write $\dfrac{37}{6}$ as a mixed number.

Steps	Reasons
$\begin{array}{r} 6r1 \\ 6\overline{)37} \\ \underline{-36} \\ 1 \end{array}$	Do long division. Write the remainder of 1 over the 6, which is called the divisor. To get the fractional part of the mixed number
$6\frac{1}{6}$	For the mixed number write $\text{quotient}\dfrac{\text{remainder}}{\text{divisor}}$

To convert mixed numbers to decimals

Keep the whole number part and divide the fraction to get the decimal part. If the fraction is a repeating decimal, identify the block that is repeating and put a line over it.

Example

3. Convert $5\frac{7}{9}$ to a decimal number.

Steps	Reasons
$5\frac{7}{9} = 5 + \frac{7}{9}$	Split the mixed number into a whole number and the fraction part.
$\begin{array}{r} .777 \\ 9\overline{)7.0000} \\ -\underline{63} \\ 70 \\ -\underline{63} \\ 70 \\ -\underline{63} \\ 7 \end{array}$	Divide the fractional part. When the remainder is the same as in a previous step in the division, the decimal part repeats.
$5.\overline{7}$	Write the whole number part, decimal point, and fractional part as a decimal. Since there is a repeating block for the decimal part, draw a line over it to show that it repeats forever.

To convert decimals to fractions

Decimals can be thought of as fractions with a whole number in the numerator (top) and the place value of the digit farthest to the right for the denominator (bottom).

Example

4. Convert 0.325 to a fraction.

Steps	Reasons
$\frac{325}{1000}$	The last digit is the thousandths. Write the decimal part as a fraction with 1000 in the denominator.
$\frac{5\cdot 5\cdot 13}{5\cdot 5\cdot 40}$	Factor and cancel the common factors.
$\frac{13}{40}$	Write the simplified fraction.

Equivalent fractions have the same value. $\frac{1}{2}, \frac{3}{6}, \frac{20}{40}, \frac{50}{100}$ are all equivalent fractions. When we multiply a fraction in the numerator and denominator by the same number, we are not changing the value of the fraction.

$\frac{1}{2} = \frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$ $\frac{3}{3}$ is one. Multiplying by one does not change the value.

$\frac{1}{2} = \frac{1}{2} \cdot \frac{20}{20} = \frac{20}{40}$ $\frac{20}{20}$ is one. Multiplying by one does not change the value.

$\frac{1}{2} = \frac{1}{2} \cdot \frac{50}{50} = \frac{50}{100}$ $\frac{50}{50}$ is one. Multiplying by one does not change the value.

To **write in simplest form** we are going in the opposite direction. We cancel the common factor.

Example

5. Write $\frac{20}{25}$ in simplest form.

Steps	Reasons
$\frac{20}{25} = \frac{4 \cdot 5}{5 \cdot 5}$	Factor numerator and denominator using the Greatest Common Factor (GCF).
$\frac{4 \cdot 5}{5 \cdot 5} = \frac{4}{5}$	Cancel the common factor, which is 5. Remember, $\frac{5}{5} = 1$
$\frac{4}{5}$	Write the simplified fraction.

Multiply fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

To multiply fractions, we multiply straight across. The numerators are multiplied to get the new numerator and the denominators are multiplied to get the new denominator. However, instead of just multiplying across it is easier to cancel any common factor in the numerator with any common factor in the denominator in the beginning.

Examples

6. Multiply: $\frac{5}{12}\cdot\frac{24}{25}$ First we show the longer way and then afterwards we will show that the short-cut gives the same answer.

Steps	Reasons
$\frac{5}{12}\cdot\frac{24}{25}$	The longer way is to start by multiplying the two numerators and multiplying the two denominators.
$\frac{5\cdot 24}{12\cdot 25}$	Multiply the numerators and denominators.
$\frac{120}{300}$	Do arithmetic.
$\frac{2\cdot 60}{5\cdot 60}$	Find the Greatest Common Factor (GCF).
$\frac{2}{5}$	When we cancel the common factor we get the answer in reduced form.

There is an easier way to multiply fractions. Instead of multiplying and then cancelling common factors, we can cancel before multiplying, which lets us cancel the numbers while they are smaller.

7. Multiply: $\frac{5}{12}\cdot\frac{24}{25}$

Steps	Reasons
$\frac{5}{12}\cdot\frac{24}{25}$	
$\frac{\overset{1}{\cancel{5}}\cdot\overset{2}{\cancel{24}}}{\underset{1}{\cancel{12}}\cdot\underset{5}{\cancel{25}}}$	24 and 12 have a common factor of 12. So, we cancel the factor in the numerator and denominator. 5 and 25 have a common factor of 5. So, we cancel the 5 in the numerator and denominator.
$\frac{2}{5}$	When we cancel the common factors, we get the answer in reduced form.

8. Find the product of $-\frac{5}{12}, \frac{1}{3}, and -\frac{8}{15}$

Steps	Reasons
$-\frac{5}{12}\left(\frac{1}{3}\right)\left(-\frac{8}{15}\right)$	Product is the result of multiplication. Since two factors are negative the product is positive.
$\frac{5}{3\cdot 4}\cdot\frac{1}{3}\cdot\frac{2\cdot 4}{3\cdot 5}$	Drop all negative signs because the product is positive when there are two negative factors. Factor so that we can cancel common factors.
$\frac{1\cdot 2}{3\cdot 3\cdot 3}$	Cancel a common factor of any numerator and any denominator. The 5´s cancel and the 4´s cancel.
$\frac{2}{27}$	Multiply numerators and multiply denominators.

The easiest way to multiply and simplify is by canceling a common factor as you go.

$$\frac{\cancel{5}^{1}}{{}_{3}\cancel{12}}\left(\frac{1}{3}\right)\left(\frac{\cancel{8}^{2}}{\cancel{15}_{3}}\right)=\frac{2}{27}$$

8 and 12 have a common factor of 4. 5 and 15 have a common factor of 5.

Dividing fractions:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c} \quad \text{or} \quad \frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}$$

To divide fractions we take the reciprocal of the divisor and multiply. That is to say we flip the fraction after the ÷ symbol and multiply (or flip the fraction in the denominator).

Examples

9. Divide $\frac{3}{8}\div\frac{9}{16}$

Steps	Reasons
$\frac{3}{8}\cdot\frac{16}{9}$	Multiply by the reciprocal of the divisor. In other words, flip the second fraction and multiply.
$\frac{3}{8}\cdot\frac{8\cdot 2}{3\cdot 3}$	Factor to cancel common factors.
$\frac{2}{3}$	3´s and 8´s cancel.

10. Find the quotient of $-3\frac{5}{11}$ and $3\frac{4}{5}$.

Steps	Reasons
$-3\frac{5}{11} \div 3\frac{4}{5}$	Quotient indicates that we are dividing.
$-\frac{38}{11} \div \frac{19}{5}$	Write the mixed numbers as improper fractions. A negative number divided by a positive number is negative.
$-\frac{38}{11} \cdot \frac{5}{19}$	Multiply by the reciprocal of the divisor. In other words, flip the second fraction and multiply.
$-\frac{2 \cdot 19}{11} \cdot \frac{5}{19}$	Factor in order to cancel the common factor of 19.
$-\frac{2}{11} \cdot \frac{5}{1}$	Cancel the common factor 19.
$-\frac{10}{11}$	The answer is a reduced proper fraction. If the answer is an improper fraction, we need to do division in order to write the answer as a mixed number. Keep the negative sign!

Deciding the order of fractions (< or >) :

1. Write equivalent fractions with the same denominator. We are looking for the least common multiple (LCM) of the denominators.
2. Compare the numerators.

Example

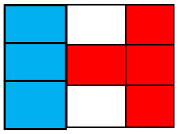
11. Place the correct symbol < or > between $\frac{7}{8} \quad \frac{13}{16}$.

Steps	Reasons
$\frac{7}{8} = \frac{7}{8} \cdot \frac{2}{2} = \frac{14}{16}$	16 is the Least Common Multiple (LCM) of the denominators. So, we write $\frac{7}{8}$ as an equivalent fraction with denominator of 16
$\frac{14}{16} > \frac{13}{16}$	The larger numerator is the larger number if the denominators are the same.
$\frac{7}{8} > \frac{13}{16}$	Write the original fractions with the correct order.

Adding fractions:

1. Find the least common denominator (or LCD) of the denominators.
2. Get the LCD for every fraction by multiplying the same numbers in the numerator (top) and denominator (bottom). That way the fractions are equivalent.
3. Add numerators and keep the common denominator.

If we add $\frac{4}{9}$ and $\frac{3}{9}$, we get $\frac{7}{9}$ as in the picture of the shaded squares:



There are seven out of nine shaded regions (or $\frac{7}{9}$) when we add the four shaded regions to the three shaded regions.

Examples

12. Add $\frac{3}{8}+\frac{1}{6}$

Steps	Reasons
$\frac{3}{8}+\frac{1}{6}$	1. Find the LCD of 8 and 6, which is 24 because 24 is the least common multiple of 8 and 6.
$\frac{3}{8}\cdot\frac{3}{3}+\frac{1}{6}\cdot\frac{4}{4}$	2. Get the LCD by multiplying the same numbers in the numerator (top) and denominator (bottom).
$\frac{9}{24}+\frac{4}{24}$	3. Add numerators and keep the common denominator.
$\frac{13}{24}$	

13. Find the sum of $-\frac{3}{4}, \frac{5}{6}, and\ \frac{2}{3}$

Steps	Reasons
$-\frac{3}{4}+\frac{5}{6}+\frac{2}{3}$	Sum means we add.
$-\frac{3}{4}\left(\frac{3}{3}\right)+\frac{5}{6}\left(\frac{2}{2}\right)+\frac{2}{3}\left(\frac{4}{4}\right)$	We need to get the Least Common Denominator (LCD), which is 12. Multiply the numerator and denominator by the same number.
$-\frac{9}{12}+\frac{10}{12}+\frac{8}{12}$	Do the multiplication in the numerator and denominator.
$\frac{-9+10+8}{12}$	Add the numerators and keep the LCD
$\frac{\cancel{9}^{3}}{\cancel{12}_{4}}$	Do the addition and simplify the fraction by cancelling the common factor of 3.
$\frac{3}{4}$	Write the answer.

14. Subtract $-\frac{3}{10}-\left(-\frac{5}{6}\right)$

Steps	Reasons
$-\frac{3}{10}+\frac{5}{6}$	Change minus a negative to addition.
$-\frac{3}{10}\left(\frac{3}{3}\right)+\frac{5}{6}\left(\frac{5}{5}\right)$	Get the LCD. By multiplying the same number in the numerator and denominator.
$-\frac{9}{30}+\frac{25}{30}$	Multiply in the numerator and the denominator
$\frac{-9+25}{30}$	Add the numerators. Keep the negative sign in the numerator. Keep the LCD
$\frac{16}{30}$	Do the addition.
$\frac{8\cdot 2}{15\cdot 2}$ or $\frac{8}{15}$	Simplify by factoring out the greatest common factor of 2 and cancel the common factor to get the answer.

15. Add $7\frac{1}{5}-4\frac{2}{3}$

Steps	Reasons
$7\frac{1}{5}-4\frac{2}{3}$	
$\frac{36}{5}-\frac{14}{3}$	Convert the mixed numbers to improper fractions: $7\cdot 5+1=36$ and $4\cdot 3+2$
$\frac{36}{5}\left(\frac{3}{3}\right)-\frac{14}{3}\left(\frac{5}{5}\right)$	Get the LCD. By multiplying the same number in the numerator and denominator.
$\frac{108}{15}-\frac{70}{15}$	Multiply in the numerator and the denominator.
$\frac{38}{15}$	Subtract the numerators and keep the LCD.
$\begin{array}{r} 2\frac{8}{15} \\ 15\overline{)38} \\ -\underline{30} \\ 8 \end{array}$	Divide to change the improper fraction to a mixed number.
$2\frac{8}{15}$	Write the answer.

For exponents and fractions just write out the multiplication.

Examples

16. Evaluate $\left(\frac{3}{4}\right)^3$

Steps	Reasons
$\left(\frac{3}{4}\right)^3$	$\frac{3}{4}$ is multiplied by itself three times.
$\left(\frac{3}{4}\right)\cdot\left(\frac{3}{4}\right)\cdot\left(\frac{3}{4}\right)$	Write out the multiplication.
$\frac{27}{64}$	Multiply the fractions by multiply the numerators and the denominators. There is no common factor to cancel.

17. Evaluate $\left(-2\tfrac{1}{5}\right)^2$

<u>Steps</u>	<u>Reasons</u>
$\left(-2\tfrac{1}{5}\right)^2$	The $-2\tfrac{1}{5}$ is multiplied by itself twice
$\left(-2\tfrac{1}{5}\right)\left(-2\tfrac{1}{5}\right)$	Write out the multiplication.
$\left(\frac{11}{5}\right)\left(\frac{11}{5}\right)$	Write the mixed numbers as improper fractions. The product of two negative numbers is positive. So, we can drop the negative signs.
$\frac{121}{25}$	Multiply the fractions.
$4\tfrac{21}{25}$	Rewrite as a mixed number by dividing.

Things can get a little tricky when we have complex fractions or lots of operations to perform. The key is to do the same operations in the same order as one would with whole numbers. Since the arithmetic operations with fractions are more complicated, you may need to work out the fractions on the side. For instance, 2+3 can be done in your head and poses no problems in the middle of a more complicated problem. However, $2\tfrac{1}{3}+3\tfrac{4}{5}$ is difficult to do mentally since it requires several steps.

When doing the next few examples it is important to keep in mind the order of operations.

The **order of operations** is as follows:

1. **Parentheses** inside to outside.
2. **Exponents.**
3. **Multiplication** and **division** together as they appear from left to right.
4. **Addition** and **subtraction** together as they appear from left to right.

Some people remember **PEMDAS** to help keep the order straight.

<u>Examples</u>

18. Simplify $\dfrac{\frac{2}{3}-\frac{3}{4}}{\frac{1}{2}+\frac{3}{5}}$

Steps	Reasons
$\dfrac{\frac{2}{3}-\frac{3}{4}}{\frac{1}{2}+\frac{3}{5}}$	The key to this problem is to treat it like three problems. First subtract in the numerator. Second add in the denominator. Third do division.
$\dfrac{\left(\frac{2}{3}-\frac{3}{4}\right)}{\left(\frac{1}{2}+\frac{3}{5}\right)}$	The numerator is one part of the fraction and the denominator is another part. So, we can place parentheses around the numerator and denominator, which helps us see the order of operations. Normally, we would not take this step to write the parentheses.
$\dfrac{-\frac{1}{12}}{\frac{11}{10}}$	First do the subtraction in the numerator. Second add in the denominator. You may do these two steps neatly to the side or at the same time that you do the subtraction in the numerator. $\begin{array}{lcl} \frac{2}{3}-\frac{3}{4} & & \frac{1}{2}+\frac{3}{5} \\ \frac{2}{3}\cdot\frac{4}{4}-\frac{3}{4}\cdot\frac{3}{3} & \text{and} & \frac{1}{2}\cdot\frac{5}{5}+\frac{3}{5}\cdot\frac{2}{2} \\ \frac{8}{12}-\frac{9}{12} & & \frac{5}{10}+\frac{6}{10} \\ -\frac{1}{12} & & \frac{11}{10} \end{array}$
	Put the fractions in the numerator and denominator. Now, there is a fraction divided by a fraction.
$-\frac{1}{12}\cdot\frac{10}{11}$	Multiply by the reciprocal of the denominator. (Flip the bottom fraction and multiply.)
$-\frac{1\cdot\cancel{2}\cdot5}{\cancel{2}\cdot6\cdot11}$	Factor and cancel the common factor.
$-\frac{5}{66}$	Multiply in the numerator and denominator.

19. Simplify $\frac{2}{3}+\frac{\frac{5}{36}}{5-\frac{1}{5}}\div\frac{5}{36}$

Steps	Reasons
$\frac{2}{3}+\frac{\frac{5}{36}}{5-\frac{1}{5}}\div\frac{5}{36}$	The key is to break this problem down into smaller problems. The first step is to simplify the complex fraction.
	Do subtraction in the denominator on the side. Just ignore $\frac{5}{36}$ until the subtraction is complete. $5-\frac{1}{5}$
$\frac{2}{3}+\frac{\frac{5}{36}}{\frac{24}{5}}\div\frac{5}{36}$	$\frac{25}{5}-\frac{1}{5}$
	$\frac{24}{5}$ When the subtraction in the denominator is finished put the $\frac{24}{5}$ back into the complex fraction.
$\frac{2}{3}+\frac{25}{864}\div\frac{5}{36}$	Do division: $\frac{\frac{5}{36}}{\frac{24}{5}}=\frac{5}{36}\cdot\frac{5}{24}=\frac{25}{864}$ Put the fraction back into the original problem.
$\frac{2}{3}+\frac{25}{864}\cdot\frac{36}{5}$	Rewrite the division as multiplication.
$\frac{2}{3}+\frac{25}{864}\cdot\frac{36}{5}$	Cancel the common factors of 36 and 5.
$\frac{2}{3}\cdot\frac{8}{8}+\frac{5}{24}$	Add fractions by getting the LCD.
$\frac{16}{24}+\frac{5}{24}=\frac{21}{24}\ or\ \frac{7}{8}$	Add numerators, keep the LCD, and then simplify if possible.

20. Evaluate $3\cdot\left(\frac{5}{6}\right)^2\cdot\left(-\frac{2}{5}\right)^3$

Steps	Reasons
$\frac{3}{1}\cdot\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdot\left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)$	Write the whole number as an improper fraction. Rewrite the fractions without exponents.
$-\frac{\cancel{3}\cdot\cancel{5}\cdot\cancel{5}\cdot\cancel{2}\cdot\cancel{2}\cdot 2}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{2}\cdot 3\cdot\cancel{5}\cdot\cancel{5}\cdot 5}$	Three factors are negative. So, the product is negative. Cancel the common factors.
$-\frac{2}{15}$	The two is left in the numerator. Multiply the remaining factors in the denominator.

Exercises:

Write as an improper fraction:

1. $3\frac{1}{2}$

2. $2\frac{3}{5}$

3. $6\frac{5}{8}$

4. $12\frac{3}{4}$

5. $-5\frac{4}{7}$

6. $-4\frac{5}{8}$

7. $-15\frac{2}{7}$

Write as a mixed number:

8. $\frac{7}{3}$

9. $\frac{23}{5}$

10. $\frac{112}{6}$

11. $\frac{77}{9}$

12. $\frac{-38}{7}$

13. $-\frac{57}{12}$

14. $-\frac{128}{15}$

Convert to a decimal number:

15. $\frac{5}{8}$

16. $\frac{9}{12}$

17. $\frac{28}{3}$

18. $-\frac{7}{11}$

19. $3\,\frac{1}{5}$

20. $8\,\frac{7}{25}$

21. $-12\,\frac{3}{16}$

22. $-14\,\frac{15}{24}$

23. $-16\,\frac{5}{9}$

24. $7\,\frac{11}{27}$

25. $32\,\frac{5}{6}$

26. $-15\,\frac{7}{11}$

Convert to a fraction:

27. 0.57

28. 0.725

29. 5.32

30. 4.75

31. 0.562

32. 0.375

Write in simplest form:

33. $\frac{12}{28}$

34. $\frac{42}{48}$

35. $\frac{125}{150}$

36. $\frac{121}{165}$

37. $\frac{84}{168}$

38. $\frac{54}{90}$

Simplify by multiplying, dividing, adding, subtracting and using the order of operations when appropriate:

39. $\frac{3}{5} \cdot \frac{4}{9}$

40. $\frac{5}{12} \cdot \frac{9}{25}$

41. $\frac{7}{27} \cdot \frac{18}{42}$

42. $-\frac{8}{27} \cdot \frac{3}{16}$

43. $\frac{30}{56} \cdot \left(-\frac{16}{45}\right)$

44. $\left(-\frac{42}{63}\right) \cdot \left(-\frac{45}{70}\right)$

45. $\left(-\frac{5}{12}\right) \cdot \left(-\frac{8}{25}\right)$

46. $3\frac{2}{3} \cdot 5\frac{1}{4}$

47. $4\frac{3}{8} \cdot 2\frac{4}{5}$

48. $\left(-1\frac{5}{8}\right)\cdot\left(-\,3\frac{5}{7}\right)$

49. $5\frac{1}{9}\cdot\left(-7\frac{2}{3}\right)$

50. $-10\frac{3}{4}\cdot 2\frac{8}{9}$

51. $\left(-6\frac{2}{7}\right)\left(-4\frac{3}{8}\right)$

52. $\frac{12}{25}\div\frac{6}{15}$

53. $\frac{9}{14}\div\frac{27}{35}$

54. $\frac{22}{42}\div\frac{33}{54}$

55. $-\frac{8}{27}\div\frac{16}{81}$

56. $\frac{25}{54}\div\left(-\frac{10}{36}\right)$

57. $-\frac{16}{25}\div\left(-\frac{15}{8}\right)$

58. $\frac{-\frac{4}{5}}{\frac{10}{11}}$

59. $\frac{\frac{4}{7}}{-\frac{12}{28}}$

60. $4\frac{2}{3}\div 2\frac{2}{5}$

61. $5\frac{5}{8}\div 3\frac{1}{3}$

62. $7\frac{2}{9}\div\left(-2\frac{6}{7}\right)$

63. $-12\frac{1}{2}\div 6\frac{2}{3}$

64. $-5\frac{1}{3}\div 2\frac{2}{7}$

65. $-3\frac{5}{6} \div \left(-11\frac{1}{2}\right)$

66. $-3\frac{7}{12} \div \left(-10\frac{6}{8}\right)$

67. $\left(\frac{2}{3}\right)^2$

68. $\left(\frac{3}{5}\right)^3$

69. $\left(-\frac{5}{7}\right)^2$

70. $\left(3\frac{1}{3}\right)^2$

71. $\left(-2\frac{3}{4}\right)^2$

72. $\left(-1\frac{3}{5}\right)^3$

73. $\frac{3}{7} + \frac{2}{7}$

74. $\frac{1}{5} + \frac{3}{5}$

75. $\frac{7}{9} - \frac{2}{9}$

76. $\frac{7}{11} - \frac{3}{11}$

77. $\frac{1}{4} + \frac{2}{3}$

78. $\frac{5}{6} + \frac{1}{12}$

79. $\frac{7}{12} - \frac{4}{15}$

80. $\frac{5}{18} - \frac{19}{30}$

81. $-\frac{5}{6} - \left(-\frac{3}{10}\right)$

82. $-\frac{5}{12}-\left(-\frac{7}{18}\right)$

83. $-\frac{11}{15}-\left(\frac{4}{25}\right)$

84. $3\frac{1}{4}-1\frac{1}{3}$

85. $2\frac{5}{6}+11\frac{2}{3}$

86. $7\frac{1}{3}+2\frac{3}{5}$

87. $4\frac{2}{9}-8\frac{5}{6}$

88. $9\frac{5}{12}-12\frac{11}{18}$

89. $3\frac{5}{6}+4\frac{3}{4}$

90. $5\frac{2}{5}+3\frac{7}{10}$

91. $3\frac{1}{6}-5\frac{2}{9}$

92. $\frac{\frac{2}{3}+\frac{1}{4}}{\frac{3}{4}-\frac{1}{6}}$

93. $\frac{\frac{5}{14}-\frac{4}{7}}{\frac{4}{21}+\frac{11}{3}}$

94. $\frac{\frac{3}{5}+\frac{1}{2}}{\frac{2}{3}-\frac{5}{6}}$

95. $\frac{\frac{1}{2}-\frac{2}{3}}{\frac{3}{5}+\frac{1}{6}}$

96. $\frac{3}{5}+\frac{4-\frac{2}{3}}{\frac{25}{9}}\div\frac{7}{2}$

97. $\frac{1}{2}+\frac{\frac{3}{8}}{\frac{5}{2}-4}\cdot\frac{2}{3}$

98. $\frac{\frac{5}{6}}{\frac{2}{3}-2}+\frac{1}{6}\cdot\frac{3}{4}$

99. $\frac{\frac{19}{5}}{3+\frac{1}{6}}-\frac{7}{8}\div\frac{3}{2}$

100. $\frac{4}{9}-\frac{3-\frac{3}{4}}{\frac{3}{4}}\div 6$

Find the answer:

101. Find the product of $-\frac{5}{6}$, $\frac{7}{15}$, and $\frac{9}{10}$.

102. Find the quotient of $5\frac{3}{4}$ and $-3\frac{1}{5}$.

103. Find the sum of $\frac{2}{3}$, $\frac{4}{9}$, and $-\frac{5}{6}$.

104. Find the difference of $\frac{7}{12}$ and $\frac{4}{15}$.

105. Find the quotient of $\frac{11}{27}$ and $\frac{5}{18}$.

106. Find the sum of $3\frac{2}{3}$, $-5\frac{7}{9}$, and $-3\frac{5}{6}$.

107. Find the difference of $-4\frac{4}{7}$ and $2\frac{3}{14}$.

108. Find the product of $-2\frac{1}{3}$, $5\frac{2}{5}$, and $-3\frac{1}{2}$.

109. If it takes $12\frac{2}{3}$ gallons of gas to fill a car’s tank, how much gas does it take to fill $\frac{3}{5}$ of the tank with gas?

110. If it takes $15\frac{3}{4}$ gallons of gas to fill a car’s tank, how much gas does it take to fill $\frac{2}{3}$ of the tank with gas?

If we square a whole number, we get what is called a perfect square:

Whole Number	Perfect Square	Whole Number Squared
0	0	0^2
1	1	1^2
2	4	2^2
3	9	3^2
4	16	4^2
5	25	5^2
6	36	6^2
7	49	7^2
8	64	8^2
9	81	9^2
10	100	10^2
11	121	11^2
12	144	12^2

If we know our multiplication table, we should be able to remember this table as well. Squaring is going from 7 to 49 because $7^2 = 49$. Taking the square root is going in the opposite direction from 49 to 7.

The <u>square root</u> of a number a is the number whose square is a.
Symbolically, c is a square root of a if $c^2 = a$

For example, the square roots of 9 are 3 and –3 because $3^2 = 9$ and $(-3)^2 = 9$.

The symbol $\sqrt{}$ refers to the positive square root. So, $\sqrt{9} = 3$. We generally will not worry about the negative square root.

<u>Examples</u>

1. Find $\sqrt{25}$

$\sqrt{25} = 5$ because $5^2 = 25$

2. Find $\sqrt{64}$

$\sqrt{64} = 8$ because $8^2 = 64$

3. Find $\sqrt{20}$

$\sqrt{20}$ is not a perfect square. So, there is no whole number square root.
We can use a calculator to get an approximation 4.4721…

For a whole number the square root is either:
1. another whole number like $\sqrt{25} = 5$ and $\sqrt{64} = 8$
2. an irrational number

An <u>irrational number</u> cannot be written as a fraction of integers. If we try to write an irrational number as a decimal number, it goes on forever without repeating, which is impossible to write. A classic example of an irrational number is π, which goes on forever without repeating. π is about 3.14159…, but if we want to indicate its exact value we need to use the symbol π. Many square roots are similar, but we can simplify many of them.

<u>Product Rule</u> for square roots:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

To simplify square roots factor out a factor that has a perfect square.

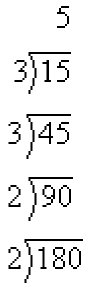
<u>Examples</u>

4. Simplify $\sqrt{20}$

Steps	Reasons
$\sqrt{20}$	Look for a factor of 20 with a square root. We can guess and check from the list of perfect squares above. Take the square root of
$\sqrt{4}\sqrt{5}$	
$2\sqrt{5}$	
$2\sqrt{5}$	Here we have the “simplified” form because we have pulled out as much as we can from underneath the square root.

5. Simplify: $\sqrt{180}$

$\sqrt{180} = \sqrt{} \cdot \sqrt{}$ The idea is to factor 180 so that one factor has a root that is a whole number and the other factor does not.



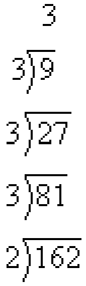
Most of us do not see that 36 is the largest factor of 180 that has a whole number root. Try the following:
1. Find the prime factorization
2. Group the pairs of factors in the prime factorization. $(2^2 \cdot 3^2)(5)$
3. These pairs of prime factors will have a square root. $(2^2 \cdot 3^2)(5) = (36)(5)$

$\sqrt{180} = \sqrt{36} \cdot \sqrt{5}$ 36 has a square root and 5 is left-over

$6\sqrt{5}$ This is the simplified form. We have pulled out as much as we can from under the square root.

6. Simplify: $5\sqrt{162}$

$\sqrt{162} = \sqrt{} \cdot \sqrt{}$ The idea is to factor 162 so that one factor has a root that is a whole number and the other factor does not.



Most of us do not see that 81 is the largest factor of 162 that has a whole number root. Try the following:
1. Find the prime factorization
2. Group the pairs of factors in the prime factorization. $(3^4)(2)$
3. These pairs of prime factors will have a square root. $(3^4)(2) = (81)(2)$

$5\sqrt{81}\cdot\sqrt{2}$ 81 has a square root and 2 is left-over

$5\cdot9\sqrt{2}$ We have pulled out as much as we can from under the square root.

$45\sqrt{2}$ Multiply the numbers.

<u>Multiplying roots</u> by using the same rule as we used to simplify: $\sqrt{a}\cdot\sqrt{b} = \sqrt{a \cdot b}$

Examples

7. Multiply $\sqrt{15}\cdot\sqrt{30}$

Steps	Reasons
$\sqrt{15}\cdot\sqrt{30}$	First multiply the parts under the square root because of the rule $\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$
$\sqrt{450}$	If you do not see that 225 is a perfect square factor of 450, finding the prime factorization may help.
$\sqrt{225}\cdot\sqrt{2}$	$450=2\cdot 3^2\cdot 5^2$ So, $3^2\cdot 5^2=225$ will have a square root. Also guessing the part of the expression that does not have a square root (2,3,5,6,7,8,…) and using your calculator will often work.
$15\sqrt{2}$	

A good trick is to multiply numbers under the root (inside the house) and multiply the numbers outside of the root (outside of the house).

8. Multiply $3\sqrt{6}\cdot 5\sqrt{10}$

Steps	Reasons
$3\sqrt{6}\cdot 5\sqrt{10}=15\sqrt{60}$	The original problem is 3 times $\sqrt{6}$ times 5 times $\sqrt{10}$. Because the only operation is multiplication we can multiply in any order (commutative property).
$=15\sqrt{4}\sqrt{15}$	Multiply under the square root (6·10=60) and multiply outside of the square root (3·5=15).
$=15\cdot 2\cdot\sqrt{15}$	Simplify square root of 60 because 4 is a perfect square factor of 60.
$=30\cdot\sqrt{15}$	

Distributive Property:

$$a(b+c)=a\cdot b+a\cdot c$$

We can use the distributive property when multiplying roots.

To add and subtract square root expressions we need to think of like terms. The root must be exactly the same. Then add or subtract the numbers in front (coefficients) and keep the same root part. It works as follows:

$5\sqrt{3}+2\sqrt{3}=7\sqrt{3}$
$5x + 2x = 7x$

Add the numbers and keep the same root part just like we do with variables.

For $3\sqrt{2}+8\sqrt{3}$ We cannot simplify further because the numbers under the square root (radicands) are different.

Examples

9. Multiply: $3\sqrt{5}\left(2\sqrt{5}+\sqrt{2}\right)$

Steps	Reasons
$3\sqrt{5}\left(2\sqrt{5}+\sqrt{2}\right)=3\sqrt{5}\cdot 2\sqrt{5}+3\sqrt{5}\cdot\sqrt{2}$	We cannot add inside the parentheses. Use the distributive property to multiply both terms inside the parentheses by the $3\sqrt{5}$, which is outside of the parentheses.
$=6\sqrt{25}+3\sqrt{10}$	
$=6\cdot 5+3\sqrt{10}$	Multiply under the square root (5·5=25 and 5·2=10). Multiply outside of the square root (3·2=6 and 3·1=3). Simplify the first root.
$=30+3\sqrt{10}$	

10. Simplify: $3\sqrt{50}-2\sqrt{18}+4\sqrt{98}$

Steps	Reasons
$3\sqrt{50}-2\sqrt{18}+4\sqrt{98}$	Since there are different numbers under the square root (radicands), we cannot collect like terms right away.
$3\sqrt{25}\sqrt{2}-2\sqrt{9}\sqrt{2}+4\sqrt{49}\sqrt{2}$	1. Simplify each root by finding a factor that is a perfect square.
$3\cdot5\sqrt{2}-2\cdot3\sqrt{2}+4\cdot7\sqrt{2}$	2. Take the square root each perfect square.
$15\sqrt{2}-6\sqrt{2}+28\sqrt{2}$	3. Multiply.
$37\sqrt{2}$	4. Now the roots are exactly the same. So, add and subtract the numbers and keep the $\sqrt{2}$

Exercises

Simplify by finding the square root. If the square root is not a whole number, use a calculator and round to the thousandth.

1. $\sqrt{49}$

2. $\sqrt{81}$

3. $\sqrt{64}$

4. $5\sqrt{121}$

5. $9\sqrt{16}$

6. $\sqrt{731}$

7. $\sqrt{157}$

8. $3\sqrt{12{,}895}$

9. $21\sqrt{45{,}693}$

Simplify by finding a factor that is a perfect square.

10. $\sqrt{75}$

11. $\sqrt{32}$

12. $\sqrt{108}$

13. $\sqrt{128}$

14. $8\sqrt{28}$

15. $5\sqrt{27}$

16. $3\sqrt{63}$

17. $15\sqrt{242}$

18. $21\sqrt{147}$

19. $\sqrt{15} \cdot \sqrt{6}$

20. $\sqrt{35} \cdot \sqrt{10}$

21. $\sqrt{30} \cdot \sqrt{70}$

22. $\sqrt{21} \cdot \sqrt{7}$

23. $7\sqrt{6} \cdot 7\sqrt{8}$

24. $4\sqrt{8} \cdot 3\sqrt{14}$

25. $7\sqrt{24} \cdot 5\sqrt{36}$

26. $5\sqrt{10} \cdot 6\sqrt{20}$

27. $\sqrt{5} \cdot \left(\sqrt{15} + \sqrt{5}\right)$

28. $\sqrt{3} \cdot \left(\sqrt{27} + \sqrt{6}\right)$

29. $\sqrt{7} \cdot \left(\sqrt{14} - \sqrt{7}\right)$

30. $\sqrt{6} \cdot \left(\sqrt{10} - \sqrt{2}\right)$

31. $3\sqrt{5} \cdot \left(2\sqrt{5} + 3\sqrt{15}\right)$

32. $4\sqrt{7} \cdot \left(2\sqrt{21} - 6\sqrt{7}\right)$

33. $7\sqrt{2} \cdot \left(5\sqrt{6} - 4\sqrt{2}\right)$

34. $6\sqrt{10} \cdot \left(3\sqrt{15} + 4\sqrt{6}\right)$

35. $4\sqrt{3} + 7\sqrt{3} - 2\sqrt{3}$

36. $8\sqrt{7} - 2\sqrt{7} + 4\sqrt{7}$

37. $3\sqrt{5}+6\sqrt{5}-7\sqrt{10}$

38. $9\sqrt{6}-11\sqrt{2}+15\sqrt{6}$

39. $12\sqrt{12}+3\sqrt{27}-4\sqrt{75}$

40. $3\sqrt{45}-12\sqrt{125}+2\sqrt{80}$

41. $7\sqrt{32}+8\sqrt{50}-10\sqrt{72}$

42. $15\sqrt{108}-12\sqrt{147}+2\sqrt{75}$

For the next two problems, the velocity (v) of a Tsunami in kilometers per hour can be modeled by the formula $\mathrm{v}=100+9.8\sqrt{\mathrm{D}}$ where (D) is the depth of the water measured in meters.

43. a. If the depth is 5000 meters, find the velocity of the tsunami in kilometers per hour to the nearest hundredth.
    b. If the depth is 600 meters, find the velocity of the tsunami to the nearest hundredth of kilometers per hour.
    c. What is the effect on the velocity of the Tsunami as the depth decreases?

44. a. If the depth is 3000 meters, find the velocity of the tsunami in kilometers per hour to the nearest hundredth.
    b. If the depth is 200 meters, find the velocity of the tsunami to the nearest hundredth of kilometers per hour.
    c. What is the effect on the velocity of the Tsunami as the depth decreases?

For the next two problems ignoring air resistance, the time (t) in seconds that it takes a dropped object to fall can be modeled by $\mathrm{t}=\sqrt{0.204\cdot\mathrm{d}}$ where the distance (d) is measured in meters.

45. From the top of the Empire state building to the ground below it is 381 meters. Ignoring air resistance, find the time that it takes for an object to fall 381 meters to the nearest tenth of a second.

46. From the top of the antenna of the Burj Khalifa in Dubai to the ground below it is 828 meters. Ignoring air resistance, find the time that it takes for an object to fall 828 meters to the nearest tenth of a second.

Exponents can be used to write repeated multiplication. $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$ The exponent of 4 tells us how many times to multiply the base, which is the 3. There are several rules for exponents, which help us to manipulate complicated exponential expressions. After each rule, there is a quick example to help show why the rule is true.

Rules for Exponents:

1. $x^m \cdot x^n = x^{m+n}$

Example: $3^2 \cdot 3^4 = (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3) = 3^6$. It is easier to add the 2 and 4 to get the 6.

2. $(x^m)^n = x^{m \cdot n}$

Example: $(5^3)^2 = (5^3)(5^3) = (5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5) = 5^6$. It is easier to multiply the exponent outside the parentheses by the exponent inside the parentheses.

3. $(x^m y^n)^p = x^{m \cdot p} y^{n \cdot p}$

Example: $(7^3 \cdot 9^4)^2 = (7^3 \cdot 9^4)(7^3 \cdot 9^4) = (7 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 9 \cdot 9)(7 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 9 \cdot 9) = 7^6 \cdot 9^8$. It is easier to multiply the exponent outside the parentheses by the exponents inside the parentheses: $3 \cdot 2 = 6$ is the exponent for 7 and $4 \cdot 2 = 8$ is the exponent for 9.

4. $\dfrac{x^m}{x^n} = x^{m-n}$

Example $\dfrac{3^6}{3^2} = \dfrac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3} = 3 \cdot 3 \cdot 3 \cdot 3 = 3^4$. It is easier to subtract 6 – 2 to get 4.

5. $x^0 = 1$

Example:

$1 = \dfrac{10^5}{10^5}$	Any number divided by itself is one. (Except for zero.)
$= 10^{5-5}$	By the rule above
$= 10^0$	Subtract 5 – 5 = 0

So, $10^0 = 1$ which works for any base except 0.

6. $x^{-n} = \dfrac{1}{x^n}$

Example:

$\dfrac{1}{5^3} = \dfrac{5^0}{5^3}$	Rule 2 says $x^0 = 1$. So, $5^0 = 1$
$= 5^{0-3}$	Rule 1.
$=5^{-3}$	Subtract 0 – 3 = –3

It should be noted that there are some restrictions here. The x and y are real numbers. There can be no zeros in the denominator.

<u>Examples</u>

1. Simplify by using the rules for exponents: $2^3 \cdot 2^5$

Steps	Reasons
$2^3 \cdot 2^5$	Use the rule $x^m \cdot x^n = x^{m+n}$
2^{3+5}	
2^8	Add the exponents because of multiplication with the same base.
256	

2. Simplify by using the rules for exponents: $\dfrac{5^9}{5^6}$

Steps	Reasons
$\dfrac{5^9}{5^6}$	Use the rule $\dfrac{x^m}{x^n} = x^{m-n}$
5^{9-6}	
5^3	Subtract the exponents because of division with the same base.
125	

3. Simplify by using the rules for exponents: $\left(3^2\right)^2$

Steps	Reasons
$\left(3^2\right)^2$	Use the rule $\left(x^m\right)^n = x^{m \cdot n}$
$3^{2 \cdot 2}$	
3^4	Multiply the exponents because of exponent outside of the parentheses.
81	

4. Simplify by using the rules for exponents: 5^{-2}

Steps	Reasons
5^{-2}	Negative exponent.
$\frac{1}{5^2}$	Just the rule 6: $x^{-n} = \frac{1}{x^n}$
$\frac{1}{25}$	Five squared is twenty-five.

5. Simplify by using the rules for exponents: -7^{-5}

Steps	Reasons
-7^{-5}	Remember that the negative sign in front is not part of the exponent. It is kept in front and is not affected by the exponent.
$-\frac{1}{7^5}$	For negative exponents use $x^{-n} = \frac{1}{x^n}$

6. Simplify by using the rules for exponents: -20^0

Steps	Reasons
-20^0	Remember that the negative sign in front is not part of the exponent. It is kept in front and is not affected by the exponent.
-1	For exponent of zero $x^0 = 1$

7. Simplify by using the rules for exponents: $\frac{4^9}{4^{11}}$

Steps	Reasons
$\frac{4^9}{4^{11}}$	First use the rule $\frac{x^m}{x^n} = x^{m-n}$
4^{9-11}	Subtract the exponents.
4^{-2}	
$\frac{1}{4^2}$	For a negative exponent, just use $x^{-n} = \frac{1}{x^n}$
$\frac{1}{16}$	Four squared is 16 in the denominator.

8. Simplify by using the rules for exponents: $\dfrac{x^8y^7}{x^3y^5}$

Steps	Reasons
$\dfrac{x^8y^7}{x^3y^5} = x^{8-3}y^{7-5}$	Subtract the exponents in the denominator (bottom) from the exponents in the numerator (top). Rule number 4.
$= x^5y^2$	Do the subtraction.

9. Simplify by using the rules for exponents: $\left(\dfrac{8x^{-5}y^7}{6x^4y^{-6}}\right)^{-2}$

Steps	Reasons
$\left(\dfrac{{}^{4}\cancel{8}x^{-5-4}y^{7+6}}{{}_{3}\cancel{6}}\right)^{-2}$	Subtract the exponent in the denominator (bottom) from the exponent in the numerator (top). $\dfrac{x^m}{x^n} = x^{m-n}$. I think of "cross the line change the sign". That way for y I can write an exponent of 7 + 6 instead of 7 – (–6). Cancel the common factor for the numbers.
$\left(\dfrac{4x^{-9}y^{13}}{3}\right)^{-2}$	Add and subtract for the exponents.
$\dfrac{4^{-2}x^{(-2)(-9)}y^{(-2)(13)}}{3^{-2}}$	Multiply the exponent outside the parentheses by all the exponents inside parentheses.
$\dfrac{3^2x^{18}y^{-26}}{4^2}$	Get rid of the negative exponents by "cross the line change the sign".
$\dfrac{9x^{18}}{16y^{26}}$	Evaluate the numbers.

10. Simplify by using the rules for exponents: $\dfrac{\left(3x^{-2}y^{7}\right)^{-2}}{\left(9x^{5}y^{-6}\right)^{-1}}$

Steps	Reasons
$\dfrac{\left(9x^{5}y^{-6}\right)^{1}}{\left(3x^{-2}y^{7}\right)^{2}}$	Flipping the numerator and denominator gets rid of the negative exponents outside of the parentheses.
$\dfrac{9x^{5}y^{-6}}{3^{2}x^{(-2)(2)}y^{7\cdot 2}}$	Multiply the exponent outside of the parentheses by the exponents inside parentheses.
$\dfrac{\cancel{9}x^{5}y^{-6}}{\cancel{9}x^{-4}y^{14}}$	Do the arithmetic in the denominator.
$x^{5+4}y^{-6-14}$	Cancel the common factor of 9. Bring the variables in the denominator up by "cross the line change the sign."
$x^{9}y^{-20}$ *or* $\dfrac{x^{9}}{y^{20}}$	Write your answer either with no variables in the denominator or no negative exponents.

Scientific notation

Large and small decimal numbers can be more conveniently written using scientific notation.

$a\times 10^{n}$ a has a digit in the ones and no digit in the tens places
a could be 3.456 or 2.3, but a could not be 27.34 or 0.56
n is an integer (…, -2,-1,0,1,2,…)

3.2×10^{7} is a big number because 10 is multiplied seven times. Multiplying by 10 moves the decimal point to the right.

$3.2\times 10^{7} = 32{,}000{,}000$ Just move the decimal point 7 times to the right.

2.4×10^{-5} is a small number. $10^{-5} = \dfrac{1}{10^{5}}$ So, we are dividing by 10 five times. We do this division by 10 moves the decimal point to the left.

$2.4\times 10^{-5} = 0.000024$ Just move the decimal point 5 times to the left.

Examples

1. First write in scientific notation and then simplify: $\frac{32,000,000,000,000,000}{160,000,000,000}$

Steps	Reasons
$\frac{32{,}000{,}000{,}000{,}000{,}000}{160{,}000{,}000{,}000}$	
$\frac{3.2\times10^{16}}{1.6\times10^{11}}$	Rewrite using scientific notation. Notice that 3.2 and 1.6 have a digit in the ones place and no digits in the tens place.
$\frac{3.2}{1.6}\times\frac{10^{16}}{10^{11}}$	We can split into two fractions or divide the decimal numbers directly and rewrite the 10's using the rule for exponents. $\frac{x^m}{x^n}=x^{m-n}$
$2\times10^{16-11}$	
2×10^{5} or $200{,}000$	Write the answer in scientific notation or as a decimal.

2. First write in scientific notation and then simplify: $\frac{(72,000,000)(0.0000000092)}{(0.000036)(1,000,000,000)}$

Steps	Reasons
$\frac{(72{,}000{,}000)(0.0000000092)}{(0.000036)(1{,}000{,}000{,}000)}$	
$\frac{(7.2\times10^{7})(9.2\times10^{-9})}{(3.6\times10^{-5})(1\times10^{9})}$	Rewrite using scientific notation. Notice that decimals have a digit in the ones place and no digits in the tens place.
$\frac{7.2\cdot9.2}{3.6\cdot1}\times\frac{10^{7}\cdot10^{-9}}{10^{-5}\cdot10^{9}}$	We can split into two fractions or divide the decimal numbers directly and rewrite the 10's using the rules for exponents:
$18.4\times\frac{10^{7+(-9)}}{10^{-5+9}}$	$x^m\cdot x^n=x^{m+n}$ add exponents in the numerator and denominator
$18.4\times\frac{10^{-2}}{10^{4}}$	$\frac{x^m}{x^n}=x^{m-n}$ subtract exponent in denominator from the exponent in the numerator.
$18.4\times10^{-2-4}$	
$1.84\times10^{1}\times10^{-6}$	To write in scientific notation requires converting the 18.4 to a decimal of 1.84, which leaves us with another 10.
1.84×10^{-5} or 0.0000184	

Exercises

Simplify by using the rules of exponents:

1. 9^2
2. 8^2
3. $(-2)^4$
4. -3^4
5. -2^4
6. $(-3)^4$
7. 5^0
8. -10^0
9. -15^0
10. 6^0
11. $(2^3)^2$
12. $(3^2)^2$
13. $2^3 \cdot 2^2$
14. $3^2 \cdot 3^3$
15. $\frac{9^7}{9^5}$
16. $\frac{4^9}{4^6}$
17. $(4^3)^2$

18. $(5^2)^2$

19. 2^{-4}

20. 6^{-2}

21. -5^{-2}

22. $(-3)^{-2}$

23. $(-4)^{-2}$

24. -4^{-3}

25. -1^{-7}

26. $4^7 \cdot 4^{-9}$

27. $3^5 \cdot 3^{-8}$

28. $\frac{5^8}{5^6}$

29. $\frac{4^9}{4^6}$

30. $\frac{3^7}{3^9}$

31. $\frac{2^5}{2^8}$

32. $3^{-8} \cdot 3^6$

33. $3^{-20} \cdot 3^{17}$

34. $\frac{2^5}{2^7}$

35. $\frac{7^9}{7^{11}}$

Simplify the following using the rules for exponents and leaving no negative exponents. Assume that none of the variables have a value of zero.

36. $x^5 \cdot x^7$

37. $y^{12} \cdot y^{-5}$

38. $z^7 \cdot z^{-11}$

39. $t^{15} \cdot t^{-20}$

40. $\frac{x^5}{x^9}$

41. $\frac{y^7}{y^{15}}$

42. $\frac{x^6 \cdot y^3}{x^2 \cdot y^{11}}$

43. $\frac{x^{10} \cdot y^7}{x^5 \cdot y^3}$

44. $\frac{x^5 \cdot y^{-3}}{x^{-8} \cdot y^7}$

45. $\frac{3^5 \cdot x^5}{3^7 \cdot x^3}$

46. $\frac{2^3 \cdot y^7}{2^6 \cdot y^9}$

47. $\left(\frac{x^5 \cdot y^7}{x^7 \cdot y^9}\right)^{-2}$

48. $\left(\frac{5^8 \cdot t^8}{5^6 \cdot t^{12}}\right)^{-2}$

49. $\left(\frac{3^4 \cdot y^5}{3^6 \cdot y^3}\right)^{-2}$

50. $\left(\frac{x^5 \cdot y^{10}}{x^9 \cdot y^6}\right)^{-2}$

51. $\frac{\left(x^3 \cdot y^7\right)^{-2}}{(x^5 \cdot y^6)^{-1}}$

52. $\frac{\left(x^4 \cdot x^{-5}\right)^{-3}}{(x^{-2} \cdot y^3)^{-2}}$

53. $\frac{\left(x^{-4} \cdot y^3\right)^5}{(x^7 \cdot y^{-5})^{-2}}$

Write the decimal number in scientific notation:

54. $49{,}000{,}000{,}000{,}000$

55. $3{,}740{,}000{,}000{,}000$

56. 0.000024

57. 0.000000007624

58. 0.00021

59. $123{,}000{,}000{,}000$

60. $94{,}370{,}000{,}000{,}000$

61. 0.0000062

First convert the decimal numbers to scientific notation and then simplify. You may leave your answer in scientific notation:

62. $156{,}000{,}000{,}000 \cdot 23{,}000{,}000{,}000$

63. $2{,}500{,}000{,}000 \cdot 234{,}000{,}000{,}000$

64. $576{,}000{,}000 \cdot 72{,}000{,}000$

65. $6{,}200{,}000{,}000 \cdot 432{,}000{,}000$

66. $\frac{735{,}000{,}000{,}000}{0.0000021}$

67. $\frac{0.00000682}{22{,}000{,}000{,}000}$

68. $\frac{(3{,}600{,}000)\cdot(0.0000022)}{(0.0012)\cdot(1{,}100{,}000{,}000)}$

69. $\frac{(0.00000028)\cdot(15{,}000{,}000)}{(350{,}000{,}000{,}000{,}000)\cdot(0.00048)}$

70. $\frac{(0.000072)\cdot(360{,}000{,}000)}{(3{,}200{,}000{,}000{,}000)\cdot(0.00081)}$

71. $\frac{(1{,}600{,}000)\cdot(0.00045)}{(0.0000012)\cdot(300{,}000{,}000)}$

For the following two problems, use 9.11×10^{-28} grams for the weight of an electron.

72. . How much do a billion electrons weigh?

73. A googol is a mathematical number with a value of 10^{100}. How much does a googol of electrons weigh?

For the next two problems, a light year is the distance that light travels in a year, which is about 5.88×10^{12} miles.

74. The furthest observed object was a gamma ray burst 13,095,000,000 light years away. How many miles away was this gamma ray burst?

75. The Andromeda galaxy is about 2,500,000 light years away. How many miles away is the Andromeda galaxy?

2

Having mastered the order of operations and signed numbers from Chapter 1 will be very important throughout this chapter. Here we begin a discussion of algebra and the algebraic techniques of simplifying expressions and solving equations. Adult students will recognize these topics and will master them with more or less effort depending on their prior experience and ability. It is not uncommon for adult students to need extra work with the more complicated problems.

Course Outcomes:

- Demonstrate mastery of algebraic skills
- Recognize and apply mathematical concepts to real-world situations

2.1 Algebraic Expressions

The notions of variable and algebraic expression are introduced. Algebraic relations are used to solve real world problems.

2.2 Simplifying Algebraic Expressions

Some properties of real numbers are defined. The idea of algebraic expression is further developed. Like terms are explained for addition and subtraction of algebraic expressions. Students will learn how to simplify complicated algebraic expressions using order of operations.

2.3 Solving Linear Equations

Linear equations are defined. Solving linear equations from simple to complex is explained.

2.4 Literal Equations

Literal equations are defined. Students will be able to solve literal equations.

<u>Algebraic expressions</u> contain numbers variables and arithmetic operations like addition, subtraction, multiplication, division, and exponents. A variable is <u>symbol</u> usually a letter that stands for many numbers or an unknown quantity.

Below algebraic expressions are evaluated for specific numbers by substituting the numbers for the variable. Use parentheses around the number that is being substituted to help avoid making some careless mistakes with operations and negative signs.

<u>Examples</u>

1. Evaluate $5x^2 + 2y$ for the $x = 3$ and $y = 7$.

<u>Steps</u>	<u>Reasons</u>
$5(3)^2 + 2(7)$	Substitute the variables with the numbers. Using parentheses, helps us see the appropriate order of operations:
$5(9) + 2(7)$	1. Evaluate exponent.
	2. Multiplication.
$45 + 14$	3. Add
59	

2. Evaluate $x + y$ for $x = 3\frac{1}{3}$ and $y = 2\frac{4}{5}$.

<u>Steps</u>	<u>Reasons</u>
$3\frac{1}{3} + 2\frac{4}{5}$	Begin with the same substitution.
$\frac{10}{3} + \frac{14}{5}$	Convert the mixed numbers to improper fractions.
$\frac{10}{3}\left(\frac{5}{5}\right) + \frac{14}{5}\left(\frac{3}{3}\right)$	Get the LCD. By multiplying the same number in the numerator and denominator.
$\frac{50}{15} + \frac{42}{15}$	Multiply in the numerator and the denominator.
$\frac{92}{15}$	Add the numerators and keep the LCD.
$\begin{array}{r} 6\frac{2}{15} \\ 15\overline{)92} \\ -\underline{90} \\ 2 \end{array}$	Divide to change the improper fraction to a mixed number.
$6\frac{2}{15}$	Write the answer.

3. Evaluate xyz for x = $-3\frac{1}{2}$, y = $-2\frac{2}{7}$, and z = $-1\frac{3}{4}$

Steps	Reasons
$-3\frac{1}{2}\left(-2\frac{2}{7}\right)\left(-1\frac{3}{4}\right)$	Substitute and write the multiplication. Note the product is negative because there are three negative factors.
$-\frac{7}{2}\cdot\frac{16}{7}\cdot\frac{7}{4}$	Change the mixed numbers to improper fractions. (Multiply whole number by denominator and add the numerator to get the new numerator. Keep old denominator.)
$-\frac{7}{2}\cdot\frac{2\cdot2\cdot4}{7}\cdot\frac{7}{4}$	Factor to cancel common factors. Any factor in any numerator cancels with any factor in any denominator.
$-\frac{1}{1}\cdot\frac{2}{1}\cdot\frac{7}{1}$	7´s, 2´s and 4´s cancel. Note that when we cancel a 1 is left.
$-\frac{14}{1}\,or\,-14$	Multiply numerators and multiply denominators.

4. Evaluate $(x+y)^2-2z$ for $x=5, y=$ – 7, z = –3

Steps	Reasons
$[(5)+(-7)]^2-2(-3)$	Replace the variables with the numbers. If the parentheses are used in parentheses are used in replacement, it is easier to determine the operation. The 5 does not need parentheses, but it is still correct to put them there.
$[-2]^2-2(-3)$	Work inside parentheses. 5 + (–7) = –2
$4-2(-3)$	Exponents: [-2]² = 4
$4+6$	Multiplication: –2(–3) = 6
10	Addition: 4 + 6 = 10

5. Evaluate $-x - (-y)$ for $x = -2$ and $y = 8$

Steps	Reason
$-(-2) - (-8)$	Replace the letter with the number using parentheses.
$2 + 8$	With two negative signs or minus a negative number, the sign becomes positive or the operation becomes addition.
10	Add the numbers.

6. Evaluate $(x + y)^2 - 3y$ for $x = 2$ and $y = 5$.

Steps	Reason
$(2 + 5)^2 - 3(5)$	1. Substitute the variables with the numbers. Using parentheses, helps us see the appropriate operations.
$(7)^2 - 3(5)$	The 2 and 5 in the parentheses do not need parentheses. 2. Perform the operations within parentheses.
$49 - 3(5)$	3. Exponents 4. Multiplication
$49 - 15$	5. Subtraction The answer is 34.
34	

7. Evaluate: $\dfrac{6x^2yz^3}{(x-y)^2}$ for $x = 4, y = -2, \textit{and } z = -3$

Steps	Reason
$\dfrac{6x^2yz^3}{(x-y)^2}$	Substitute the variables with the given values for the variables.
$\dfrac{6(4)^2(-2)(-3)^3}{(4-(-2))^2}$	Use the order of operations to simplify. Treat the numerator (top) and denominator (bottom) separately: In the numerator first do exponents then multiply.
$\dfrac{6(16)(-2)(-27)}{(4+2)^2}$	In the denominator work inside the parentheses then evaluate the exponent.
$\dfrac{5184}{6^2}$	
$\dfrac{5184}{36} \textit{ or } 144$	Simplify the fraction.

Exercises

1. Evaluate $x + y^2$ for $x = 3$ and $y = 2$.
2. Evaluate $x^3 - y^2$ for $x = 5$ and $y = 3$.
3. Evaluate $3x + y^2$ for $x = -3$ and $y = -2$.
4. Evaluate $5x + y^2$ for $x = -4$ and $y = -3$.
5. Evaluate $7x^2 - 3y$ for $x = -3$ and $y = 2$.
6. Evaluate $x + 3y$ for $x = \frac{2}{3}$ and $y = -\frac{1}{2}$.
7. Evaluate $3x - y^2$ for $x = -\frac{3}{4}$ and $y = -\frac{2}{5}$.
8. Evaluate xyz for $x = 4$, $y = -5$, and $z = -2$.
9. Evaluate $2xyz$ for $x = -2$, $y = -3$, and $z = -7$.
10. Evaluate $-3xyz$ for $x = -5$, $y = -1$, and $z = -2$.
11. Evaluate $\frac{5x+2z}{3x-2y}$ for $x = -2$, $y = 3$, and $z = -7$.
12. Evaluate $\frac{3x-2y}{xy}$ for $x = 2$, $y = -1$.
13. Evaluate xyz for $x = -\frac{3}{4}$, $y = \frac{5}{9}$, and $z = -\frac{8}{15}$.
14. Evaluate $-2xyz$ for $x = \frac{5}{12}$, $y = -\frac{9}{25}$, and $z = \frac{10}{27}$.
15. Evaluate $-3xyz$ for $x = -\frac{3}{4}$, $y = \frac{5}{9}$, and $z = -\frac{8}{15}$.
16. Evaluate $4xy^2$ for $x = -2\frac{1}{3}$, and $y = 1\frac{2}{3}$.
17. Evaluate $2x^2y$ for $x = 2\frac{1}{2}$, $y = -3\frac{3}{5}$.
18. Evaluate $4x^2y^2 - 2xy + 3x^2y^2$ for $x = 4$, $y = 5$.

19. Evaluate $2x^2y^2 + xy + 3y^2$ for $x = 3$, $y = 2$.

20. Evaluate $\frac{2xyz}{2x-y-z}$ for $x = -5$, $y = 2$, and $z = -4$.

21. Evaluate $\frac{3x-2y}{xy}$ for $x = 2$, $y = -1$.

22. Evaluate $x^2y^2 - 5xy + x^2y^2$ for $x = -2$, $y = -3$.

23. Evaluate $5x^2y^2 - 2xy + 4x^2y^2$ for $x = -3$, $y = -1$.

24. Evaluate $3x^2y^3 - 2xy - x^3y^2$ for $x = -2$, $y = 3$.

25. Evaluate $x^2y^2 - 3xy - 2x^2y^2$ for $x = -1$, $y = -3$.

26. The cost of a house in dollars can be estimated by the formula $C = 7100x + 27{,}500$ where x is the number of years after 1970.
   a. Use the formula to estimate the cost of a house in 2005.
   b. If the actual cost of a house in 2005 was $280,000, does the formula underestimate or overestimate the actual price? By how much?

27. People's heights have been increasing over the last three hundred years. The formula $H = .03t + 62$ can be used to estimate men's average heights in inches t years after 1700.
   a. Use the formula to estimate men's average height in the year 2010.
   b. If the actual average height of men is 71 inches, does the formula underestimate or overestimate the actual height? By how much? Do you think that this trend will continue indefinitely?

28. The cost of a new car in dollars can be estimated by the formula $C = -1.8x^2 + 975x + 9500$ where x is the number of years after 1990.
   a. Use the formula to estimate the cost of a new car in 2015.
   b. If the actual cost of a new car in 2015 was $31,950, does the formula underestimate or overestimate the actual price? By how much?

29. The number of students at a large university can be estimated by the formula $N = -5x^2 + 800x + 4000$ where x is the number of years after 1980.
   a. Use the formula to estimate the number of students in the year 2000.

b. If there were 17,800 students in the year 2000, does the formula underestimate or overestimate the actual number of students? By how much?

Many of the properties of real numbers we know even if we do not know their name. For instance, the Multiplication Property of Zero tells us that if we multiply by zero we get zero. We knew that! We need to know the names of the properties in order to communicate why we are able to perform certain steps. The Commutative property, Associative Property, and Distributive Property are used frequently when simplifying algebraic expressions.

The Commutative Property states we can add or multiply in either order.

$a + b = b + a$ addition
$a \cdot b = b \cdot a$ multiplication

For example, 5+3=3+5 shows commutative property of addition

The Associative Property allows us to change the grouping if we have all addition or all multiplication.

$(a + b) + c = a + (b + c)$ addition
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ multiplication

An example showing the commutative property of multiplication:

$$\begin{aligned} (2\cdot 3)\cdot 4 &= 2\cdot(3\cdot 4) \\ (6)\cdot 4 &= 2\cdot(12) \\ 24 &= 24 \end{aligned}$$

Between the Commutative Property and the Associative Property:
1. If there is only addition, we can add in any order. Subtraction can be thought of as adding a negative number.
2. If there is only multiplication we can multiply in any order.

It is especially useful to think of these two properties like this when we have variables:
1. $2 + 3x + 5 + 7x = 7 + 10x$ I just add the numbers and add the variables.
2. $(–3)x(– 2) = 6x$ I just multiply the numbers. I can change the order mentally.

The Distributive Property lets us change the order of operations. We multiply outside parentheses before doing addition or subtraction inside parentheses.

$2 (3 + 5) = 2\cdot 3 + 2\cdot 5$ Here we use the distributive property. Both 3 and 5 are multiplied by the 2 outside of parentheses

We see that the distributive property works by simplifying both sides.

2 (3 + 5)	2·3 + 2·5	
2(8)	6+10	
16	16	Both sides are the same.

The Distributive Property is important with variables because we cannot always do the addition or subtraction in the parentheses because we do not always have like terms.

$3(x + 5) = 3 \cdot x + 3 \cdot 5$ We cannot add x + 5 in parentheses because they are not like terms; however, we can do the multiplication by the distributive property.

$= 3x + 15$ Here we have simplified the expression.

Examples

1. Simplify $3x(-5)$

$3x(-5) = -15x$ Done. The answer is –15x.

We want to do this problem in one step. Just multiply the numbers and keep the variable. Because it is all multiplication, we can multiply in any order. We multiply the numbers and get -15. Why does this work?

$3x(-5) = 3(-5)x$ Use the commutative property to change the order for multiplication.

$= -15x$ $3(-5) = -15$

2. Simplify $\left(-\frac{2}{3}\right)x\left(-\frac{3}{2}\right)$

$\left(-\frac{2}{3}\right)x\left(-\frac{3}{2}\right) = x$ Done. The answer is x.

Multiply the numbers $\left(-\frac{2}{3}\right)\left(-\frac{3}{2}\right) = 1$. We get $1x$ or x. The key to working with variable is focusing on the appropriate arithmetic. We can multiply the original expression in any order because it as all multiplication (commutative property). Just multiply the numbers and keep the variables.

3. Simplify $9 - 15m + 15m$

$9 - 15m + 15m = 9$ Done. The answer is 9.

$-15m + 15m = 0$ We are adding the same numbers but with opposite signs.

4. Simplify $3(2x-5y+6)$

$3(2x-5y+6)=3\cdot 2x-3\cdot 5y+3\cdot 6$ Distributive Property

$=6x-15y+18$ We can do this problem in one step.

This problem can be done in one step because the arithmetic is easy. We always need to be careful with our signs and multiply each term inside the parentheses by the two which is outside the parentheses.

5. Simplify $-5(5m-7n-11)$

$-5(5m-7n-11)=-25m+35n+55$ **Done. The answer is –25m+ 35n + 55.**

Multiply each number inside parentheses by the -5 outside parentheses. The signs can be tricky at first. Just remember when multiplying, two negative factors yield a positive product. If only one factor is positive, then the product is negative.

The missing steps are as follows:

$-5(5m-7n-11)=-5(5m)-(-5)7n-(-5)11$ Distributive Property

$=-25m+35n+55$ Multiply numbers. Careful with signs.

6. Rewrite $-(6s-5t-36)$ without parentheses.

$-(6s-5t-36)=-6s+5t+36$ Done. The answer is –6s + 5t + 36.

When there is a negative sign outside of parentheses change all the signs inside the parentheses. Easy, remember it, done.

Lets look at the steps to see why it is true. In practice, we just change all the signs in one step like we did above.

$-(6s-5t-36)=-1(6s-5t-36)$ –3 = –1·3. So, we can replace the negative sign with negative one.

$(-1)(6s)-(-1)(5t)-(-1)(36)$ Distributive property

$-6s+5t+36$ Multiply and – (–) becomes +

For collecting like terms, the variable part must be exactly the same.

$3\mathbf{x} + 2\mathbf{x} = 5\mathbf{x}$ We add the numbers and keep the variable part.

$7\mathbf{x}^2\mathbf{y} - 4\mathbf{x}^2\mathbf{y} = 3\mathbf{x}^2\mathbf{y}$ Because the variable part is exactly the same, we subtract the numbers and keep the same variable part.

$12\mathbf{x} - 5\mathbf{y}$ We stop. We cannot go further because the variable part is not the same.

$6\mathbf{x}\mathbf{y}^2 - 3\mathbf{x}^2\mathbf{y}$ We stop. We cannot go further because the variable part is not the same. The exponents are with different variables.

$11\mathbf{x}^2 + 3\mathbf{x}$ We stop. We cannot go further because the variable part is not the same. One of the x's has exponent 2 and the other does not.

The key to adding and subtracting with variables is that the variable part must be exactly the same. If the variable part is not the same, then we cannot add or subtract. Also, the variable part stays the same when we add and subtract.

Examples

7. Simplify $12t^2 + 5t - 7 + 11t^2 - 8t + 9$

Steps	Reasons
$12t^2 + 5t - 7 + 11t^2 - 8t + 9$	For t^2, $12t^2 + 11t^2 = 23t^2$ For $t, 5t - 8t = -3t$
$23t^2 - 3t + 2$	For the numbers, $-7 + 9 = 2$
	Done. The answer is $23t^2 - 3t + 2$.

This problem should be done in one step. Add or subtract the numbers in front of the t^2, t, and the numbers. We can add the numbers in any order because of the Commutative Property. Even the subtraction can be thought of as addition by adding a negative number.

8. Simplify $5x - 3(2x + 6) + 4(7x - 8)$

Steps	Reasons
$5x - 3(2x + 6) + 4(7x - 8)$	Nothing can be done in parentheses.
$5x - 6x - 18 + 28x - 32$	Use the distributive property to get rid of parentheses. Here distribute –3 and 4.
$27x - 50$	Collect like terms.

9. Simplify: $12z^2 - 5[3(2z-4)-6(z^2+1)]$

<u>Steps</u>	<u>Reasons</u>
$12z^2 - 5\ [3\ (2z-4)-6\ (z^2+1)]$	Start inside the brackets.
$12z^2 - 5\ [\ 6z - 12 - 6z^2 - 6]$	Distribute the 3 and the – 6.
$12z^2 - 5\ [\ 6z - 18 - 6z^2\ ]$	Collect like terms in the brackets. $-12 - 6 = -18$
$12z^2 - 30z + 90 + 30z^2$	Distribute the –5.
$42z^2 - 30z + 90$	Collect the like terms. $12z^2 + 30z^2 = 42z^2$

10. Add: $(3x^2 - 2x + 5) + (7x^2 + 2x - 4)$

<u>Steps</u>	Reasons
$(3x^2 - 2x + 5) + (7x^2 + 2x - 4)$	To add the two expressions just collect like terms.
$3x^2 - 2x + 5 + 7x^2 + 2x - 4$	Drop the parentheses because there is nothing to distribute.
$10x^2 + 1$	Collect like terms: $3x^2 + 7x^2 = 10x^2$ $-2x + 2x = 0$ $5 - 4 = 1$

11. Subtract: $(8x^3 - 5x + 11) - (4x^3 - 6x^2 + 5x + 11)$

<u>Steps</u>	<u>Reasons</u>
$(8x^3 - 5x + 11) - (4x^3 - 6x^2 + 5x + 11)$	Before subtracting we must change all the signs in the parentheses after subtraction.
$8x^3 - 5x + 11 - 4x^3 + 6x^2 - 5x - 11$	Now we collect like terms.
$4x^3 + 6x^2 - 10x$	Collect like terms: $8x^3 - 4x^3 = 4x^3$ $6x^2$ $-5x - 5x = -10x$ $11 - 11 = 0$

Exercises

Simplify the algebraic expression:

1. $-5 \cdot x \cdot y \cdot 2$

2. $7 \cdot t \cdot z \cdot (-3)$

3. $\left(\frac{3}{5}\right) xy \left(\frac{5}{3}\right)$

4. $\left(-\frac{4}{7}\right) rs \left(\frac{7}{4}\right)$

5. $12 - 2x + 5x$

6. $7x + 3 - 2x + 4$

7. $15 - 3y - 7y - 25$

8. $0.25x - 3.21 + 2.45x + 5.2$

9. $3.21t - 5.35 - 4.76t - 3.27$

10. $3x - 5y + 4z - 2x - 7y - 8z$

11. $5 - 3x + 2y - 7 + 8x - 9y$

12. $-1.32x + 0.32y - 3.83 + 5.21x - 5.31y - 2.73$

13. $4.21x - 9.8y - 12.76 - (-2.13x) + 12.3y - 5.35$

14. $3(2x - 5y)$

15. $4(5x - 2y + 3)$

16. $-3(3x + 2y - 4)$

17. $-5(4x - 3y - 8)$

18. $-(9x - 12y - 5)$

19. $-(2x - 7y + 8)$

20. $3x + 4x - 3y + 2y$

21. $-5x + 8x - 7y - 8y$

22. $12m - 3m + 8n - 15n$

23. $7s - 15s - 11t - 3t$

24. $5x^2 - 3x^2 - 5x + 4x$

25. $9x - 5x^2 + 2x - 3x^2$

26. $5x - 8x^2 + 3x - 10x^2$

27. $7x^2 + 2x^2 - 9x - 3x$

28. $-7xy^2 + 8x^2y - 9xy^2 + 5x^2y$

29. $12x^2y - 3xy^2 - 15x^2y + 10xy^2$

30. $12x + 5 + 2(3x - 7)$

31. $5x - 3 - 4(2x - 8)$

32. $7x - 8 - 3(5x - 4)$

33. $3x - 4 + 5(2x + 1)$

34. $8x - 3 - (3x - 5)$

35. $7x + 4 - (2x - 7)$

36. $4x + 5 - (4x + 1)$

37. $3x - 9 - (2x - 5)$

38. $12x - 9 - (5x - 7) + 4$

39. $15x - 12 - (4x + 9) - 8$

40. $(4x^2 + 2x - 3) + (3x^2 - 4x - 3)$

41. $(2x^2 - 3x + 8) + (2x^2 + 3x - 6)$

42. $(3x^2 - 5x - 7) - (2x^2 + 4x - 9)$

43. $(7x^2 - 3x + 2) - (5x^2 - 3x + 8)$

44. $(4x^2 - 3x + 2) + (x^2 + x - 3)$

45. $(3x^2 + 2x - 5) + (x^2 - 7)$

46. $(2x^2 - 3x) - (x^2 - 5x - 4)$

47. $(5x^2 - 3x - 9) - (x^2 - 5x - 9)$

48. $(4x^2 - 3x + 8) + (2x^2 + 4x - 11)$

49. $(3x^2 - 8x - 9) + (4x^2 + 8x - 9)$

50. $7 - 2[3(2x - 5) - (5x + 3)]$

51. $5 - 3[2(3x + 1) - (2x - 3^2)]$

52. $6{+}4[-2(5x + 2^2) + 3(7x - 5^2)]$

53. $8 + 2[-4(4x - 3^2) + 2(3x - 2^2)]$

54. $5 - 3[3(2x - 3^2) - (4x - 2^2)]$

55. $4 - 5[2(5x - 4^2) - (12x - 3^2)]$

56. $2\{5 - [3(2x - 2^2) - (-3x) + 2(5x - 2^2)]\}$

57. $3 - 2\{7x - [-2(3x - 1) + 3(5x - 3^2)]\}$

58. $5t^2 - 3\{-4t^2 + 2[(3t^2 - 2t + 1) - (5t^2 - 2t - 3^2)]\}$

59. $4y^2 - 3\{2y^2 - [(4y^2 - y + 2) - (3y^2 + 2y - 5)]\}$

A **linear equation** can be written in the form ax+b=0 where a,b are numbers and x is a variable. Many times we will have to simplify before solving these equations, but we should recognize that there is one variable, which may appear more than once, the variable does not have an exponent (nor is it under a root), and the variable is not in the denominator.

Solutions to equations are any true replacement for the variable. We can check to see if $x=3$ is a solution to the equation $5x-2=3x+4$ by replacing the x's with 3 in the equation.

$$5x-2=3x+4$$
$$5(3)-2=3(3)+4$$
$$15-2=9+4$$
$$13=13$$

Substitute x with 3 and do the arithmetic.

Since we end up with a true statement $13=13$, $x=3$ is a solution to the equation $5x-2=3x+4$.

At the heart of solving equations are two basic properties:
1. We can add or subtract the same number from both sides of an equation.
2. We can multiply and divide the same number on both sides of an equation.

When solving these linear equations, the objective is to isolate the variable.

The basic steps:
1. Determine how the variable is connected to the number.
2. Perform the opposite operation on both sides of the equation.
3. Check by replacing the solution for the variable in the original equation.

Examples

1. Solve $x+15=34$

Steps	Reasons
$x+15=34$	To solve the equation isolate the variable *x*.
$-15 \quad -15$	Because *x* added to 15, subtract 15 from both sides. Since 15 – 15 = 0, the *x* is left alone.
$x=19$	Here the equation is solved. The variable appears alone.

Notice that the 15 went to the other side by changing the + 15 to – 15.

To check replace the solution in the original equation. $19+15=34$ is a true statement.

2. Solve $4x = 36$

Steps	Reasons
$\frac{4x}{4} = \frac{36}{4}$	The variable *x* needs to be isolated. Because the 4 is connected to the *x* by multiplication, both sides of the equation are divided by 4.
$x = 9$	The variable is isolated and the equation is solved.

Check: 4(9) = 36 is true. In the end we just divided on the right and got rid of the 4 on the left.

3. Solve $-4x = 32$

Steps	Reasons
$-4x = 32$	The –4 is connected to the variable by multiplication. So, divide both sides by –4.
$\frac{-4x}{-4} = \frac{32}{-4}$	Divide both sides by –4. The x will be left alone because (–4) ÷ (–4) = 1 and 1x is the same as x.
$x = -8$	All done.

Check your answer by putting –8 into the original equation:
$-4(-8) = 32$ is true.

The numbers may be fractions. You should still determine how the number is connected to the variable and perform the opposite operation

4. Solve $\frac{2}{3}x = 6$

Steps	Reasons
$\frac{2}{3}x = 6$	We want to get the variable by itself. The fraction is connected to the variable by multiplication. We should divide both sides by $\frac{2}{3}$.
$\frac{3}{2} \cdot \frac{2}{3}x = 6 \cdot \frac{3}{2}$	Dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{3}{2}$.
$x = \frac{6}{1} \cdot \frac{3}{2}$	Write 6 as an improper fraction.
$x = 9$	Cancel the common factor.

5. Solve $-\frac{3}{5}=t+\frac{1}{2}$

Steps	Reasons
$-\frac{3}{5}=t+\frac{1}{2}$	We want to get the variable by itself. The fraction is connected to the variable by addition. We should subtract both sides by $\frac{1}{2}$.
$-\frac{3}{5}=t+\frac{1}{2}$ $-\frac{1}{2} \quad -\frac{1}{2}$	Subtracting on the right yields t. To subtract on the left requires more steps. Go to the side and do the necessary steps. $-\frac{3}{5}-\frac{1}{2}=-\frac{3}{5}\cdot\frac{2}{2}-\frac{1}{2}\cdot\frac{5}{5}=\frac{-6}{10}-\frac{5}{10}=\frac{-11}{10}\, or\, -1\frac{1}{10}$
$t=-1\frac{1}{10}$	Write your answer with the variable on the left.

Here we are starting to solve more complicated equations of the form ax + b = c. The idea is still to get x alone. Now we will need to perform two operations in order to get the x alone.

Steps:
1. Add or subtract from both sides.
2. Multiply or divide both sides.

Note: We tend to add and subtract first. This is different from the order of operations. We do not want to divide first because we may introduce more fractions.

Examples

6. Solve: $3x-5=7$

Steps	Reasons
$3x-5=7$	Get the x alone. Notice if we divide both sides by 3 we are going to get fractions.
$3x-5=7$ $+5 \quad +5$	Add 5 to both sides.
$3x=12$	Do the addition.
$\frac{3x}{3}=\frac{12}{3}$	3 is connected to the variable by multiplication. So divide both sides by 3.
$x=4$	Do the division to get the solution.

Check:
Replace the variable in the original equation with the solution. Use parentheses

$3(4)-5=7$

$12-5=7$

$7=7$

7. Solve: $8=5+6z$

Steps	Reasons
$8=5+6z$ $-5 \quad -5$	Get the z alone. Notice if we divide both sides by 6 we are going to get fractions. So, begin by subtracting 5 from both sides.
$3=6z$	Do the subtraction.
$\frac{3}{6}=\frac{6z}{6}$	6 is connected to the variable by multiplication. So divide both sides by 6.
$\frac{1}{2}=z$ or	Simplify the fraction.
$z=\frac{1}{2}$	Write the answer with the variable on the left.

8. Solve: $\frac{2}{3}x-\frac{4}{5}=-\frac{3}{4}$

Steps	Reasons
$\frac{2}{3}x-\frac{4}{5}=-\frac{3}{4}$	Get the x alone. We still need to add and multiply both sides, but this time there are fractions.
$\frac{2}{3}x-\frac{4}{5}=-\frac{3}{4}$ $+\frac{4}{5} \quad +\frac{4}{5}$	$\frac{4}{5}$ is subtracted. So add $\frac{4}{5}$ to both sides of the equation.
$\frac{2}{3}x=\frac{1}{20}$	Add the fractions on the side: $-\frac{3}{4}+\frac{4}{5}=-\frac{3}{4}\cdot\frac{5}{5}+\frac{4}{5}\cdot\frac{4}{4}=\frac{-15}{20}+\frac{16}{20}=\frac{1}{20}$
$\frac{3}{2}\cdot\frac{2}{3}x=\frac{1}{20}\cdot\frac{3}{2}$	Instead of dividing by $\frac{2}{3}$ we save a step by immediately multiplying by $\frac{3}{2}$.
$x=\frac{3}{40}$	Simplify by doing the multiplication.

Check:

Replace the variable in the original equation with the solution. Use parentheses

$$\frac{2}{3}\left(\frac{3}{40}\right)-\frac{4}{5}=-\frac{3}{4}$$
$$\frac{1}{20}-\frac{4}{5}=-\frac{3}{4}$$
$$\frac{1}{20}-\frac{4}{5}\cdot\frac{4}{4}=-\frac{3}{4}$$
$$\frac{1}{20}-\frac{16}{20}=-\frac{3}{4}$$
$$-\frac{15}{20}=-\frac{3}{4}$$
$$-\frac{3}{4}=-\frac{3}{4}$$

Same answer both sides checks.

9. Solve: $7x-15-10x=6$

Steps	Reasons
$7x-15-10x=6$	Here we can simplify the left before we start.
$-3x-15=6$	Collect like terms.
$-3x-15=6$ $\quad +15 \quad +15$	15 is subtracted. So add 15 to both sides.
$-3x=21$	Add.
$\frac{-3x}{-3}=\frac{21}{-3}$	–3 is multiplied by x. So divide both sides by –3.
$x=-7$	Do the division.

To check replace the variable in the original equation with the solution. Use parentheses

$$7(-7)-15-10(-7)=6$$
$$-49-15+70=6$$
$$-64+70=6$$
$$6=6$$

When working with more complicated equations, we may need to simplify one or both sides of the equation by collecting like terms.

10. Solve: $5x-1+2x=4x+8$

Steps	Reasons
$5x-1+2x=4x+8$	Start by collecting like terms on the left hand side of the equation.
$7x-1=4x+8$	Adding variables: $5x+2x=7x$
$7x-1=4x+8$ $-4x \qquad -4x$	I chose to get variables on the left. So I get rid of the 4x on the right by subtracting it from both sides.
$3x-1=8$	Subtracting variables: $7x-4x=3x$
$3x-1=8$ $+1 \quad +1$	Now it is like section 6.2 Get the numbers on the right. 1 is subtracted. So we add 1 to both sides.
$3x=9$	Add.
$\frac{3x}{3}=\frac{9}{3}$	Because x is connected to the variable by multiplication we divide both sides by 3.
$x=3$	Divide.

Check:
Replace the variable in the original equation with the solution. Use parentheses

$$5(3)-1+2(3)=4(3)+8$$
$$15-1+6=12+8$$
$$20=20$$

<u>These example demonstrates all the possible steps for solving linear equations:</u>
1. Simplify both sides of the equation by collecting like terms.
2. Get all the variables on one side of the equation and all the numbers on the other side of the equation. Here we do the opposite operation of what we see.
3. Get the variable alone by dividing both sides of the equation by the number in front of the variable.

Sometimes students confuse steps 1 and 2. In step 1 we treat each side separately and perform the operation as we see it. In step 2, we perform the opposite operation of what we see to get the variables or numbers to the other side.

11. Solve: $3-5(x+3)=3(x+2)+14$

Steps	Reasons
$3-5(x+3)=3(x+2)+14$	Simplify both sides by performing using the order of operations.
$3-5x-15=3x+6+14$	Multiply using the distributive property to get rid of the parentheses on both sides.
$-12-5x=3x+20$	Add and subtract the like terms on each side.
$-12-5x=3x+20$ $+12 \qquad\qquad +12$	Here I decided to get the variables on the left and the numbers on the right. Now I am doing the opposite operation. The –12 becomes + 12. We use the opposite operation to solve equations.
$-5x=3x+32$	Add on both sides.
$-5x=3x+32$ $-3x \quad -3x$	Get the variables on the left by using the opposite. 3x is positive. So subtract 3x from both sides.
$-8x=32$	Subtract on both sides.
$\frac{-8x}{-8}=\frac{32}{-8}$	–8 is connected to x by multiplication. Divide both sides by –8.
$x=-4$	Divide.

Check:

Replace the variable in the original equation with the solution.

$$3-5(-4+3)=3(-4+2)+14$$
$$3-5(-1)=3(-2)+14$$
$$3+5=-6+14$$
$$8=8$$

12. Solve: $2x-3(2x+1)=2(3-5x)+9$

Steps	Reasons
$2x-3(2x+1)=2(3-5x)+9$	Simplify both sides by performing using the order of operations.
$2x-6x-3=6-10x+9$	Multiply using the distributive property to get rid of the parentheses on both sides.
$-4x-3=-10x+15$	Add and subtract the like terms on each side.
$-4x-3=-10x+15$ $+10x \qquad +10x$	Here I decided to get the variables on the left and the numbers on the right. Now I am doing the opposite operation. The –10x becomes + 10x. We use the opposite operation to solve equations.
$6x-3=15$	Add on both sides.
$6x-3=15$ $+3 \quad +3$	Get the number on the right by using the opposite. 3 is subtracted. So add 3 to both sides.
$6x=18$	Add on both sides.
$\frac{6x}{6}=\frac{18}{6}$	6 is connected to x by multiplication. Divide both sides by 6.
$x=3$	Divide.

Check:

Replace the variable in the original equation with the solution. Use parentheses

$$2(3)-3(2(3)+1)=2(3-5(3))+9$$
$$2(3)-3(6+1)=2(3-15)+9$$
$$2(3)-3(7)=2(-12)+9$$
$$6-21=-24+9$$
$$-15=-15$$

Exercises

Solve and check.

1. $x - 3 = 7$

2. $x + 5 = 11$

3. $x + 9 = -5$

4. $x - 12 = -15$

5. $x - \frac{2}{3} = \frac{3}{4}$

6. $x + \frac{1}{2} = \frac{3}{5}$

7. $x + 0.25 = 0.13$

8. $x - 1.34 = -2.19$

9. $x + 2\frac{3}{4} = 1\frac{7}{8}$

10. $x - 3\frac{1}{3} = -4\frac{5}{6}$

11. $5x = -35$

12. $3x = -21$

13. $-4x = -16$

14. $-6x = -18$

15. $4.2x = -14.7$

16. $-6.4x = -4.8$

17. $\frac{2}{3}x = -\frac{4}{7}$

18. $-\frac{3}{4}x = -\frac{9}{16}$

19. $4x - 3 = 13$

20. $2x + 9 = 17$

21. $3x - 7 = -19$

22. $5x + 16 = -14$

23. $7 - 2x = 3$

24. $15 - 4x = -9$

25. $-11 - 3x = 4$

26. $14 - 6x = -10$

27. $3(x - 4) = 5(x + 4)$

28. $4(3x - 3) = 2(2x + 6)$

29. $5(2x + 3) = 3(2x + 4) - 13$

30. $-2(3x - 5) = 4(x - 5)$

31. $-3(x - 5) = 6 - 4(2x - 1)$

32. $5 - (x - 3) = 16 - 3(x + 4)$

33. $11 - (3x - 7) = 12 - 5(x + 2)$

34. $14 - 2(3x + 6) = 3x - (2x - 16)$

35. $27 - 3(x + 4) = 4x - (2x - 20)$

36. $15 - (3x - 8) = 4x - 5(3x - 7) - 25$

37. $11 - [2x - (-7)] = -6 - 4(3x - 8)$

38. $16 - [4x - (-5) - 8] = -5[2x - (-7) - 5] - 9$

39. $2[\mathrm{x}-(-3\mathrm{x})-(-5)-7]=-3[2\mathrm{x}-11-(-7)]+8$

40. $-5[2\mathrm{x}-(-4\mathrm{x})-(-3)+7]=-3[2-5\mathrm{x}-(-3\mathrm{x})-7]+12\mathrm{x}$

Literal equations have more than one variable. Literal equations may be formulas in one form that we want to manipulate to get into another form. The key is to focus on the variable that you are solving for and get everything else on the other side as if you were solving an equation with one variable.

Examples

1. Solve $P = 2L + 2W$ for W.

Steps	Reasons
$P = 2L + 2W$	We want W alone:
$-2L \quad -2L$	1. Get the 2L on the other side by subtracting it from both sides.
$P - 2L = 2W$	
$\frac{P-2L}{2} = \frac{2W}{2}$	2. Divide both sides by 2 to get W by itself.
$W = \frac{P-2L}{2}$	

2. Solve $2x - 5y = 20$ for y.

Steps	Reasons
$2x - 5y = 20$	We want y alone:
$-5y = -2x + 20$	1. Get the 2x on the other side by subtracting it from both sides.
$\frac{-5y}{-5} = \frac{-2x+20}{-5}$	2. Divide both sides by –5 to get y by itself.
$y = \frac{-2x}{-5} + \frac{20}{-5}$	3. Simplify the fractions.
$y = \frac{2}{5}x - 4$	

3. Solve $A = P + Pmt$ for t

Steps	Reasons
$A = P + Pmt$	
$A - P = Pmt$	We want to solve for t. First we need to subtract P from both sides.
$\frac{A-P}{Pm} = \frac{Pmt}{Pm}$	Next divide both sides by Pm.
$\frac{A-P}{Pm} = t$	We get t alone on the right.
$t = \frac{A-P}{Pm}$	Generally we write our answer with the variable we are solving for on the left.

4. Solve $A = \frac{x+y+z}{3}$ for x.

Steps	Reasons
$A = \frac{x+y+z}{3}$	If there is a number or variable in the denominator, clear fractions.
$3 \cdot A = \frac{x+y+z}{\cancel{3}} \cdot \frac{\cancel{3}}{1}$	1. Multiply both sides by the denominator 3.
$3A = x + y + z$	2. Subtract y and z from both sides to get the x alone.
$3A - y - z = x$	
$x = 3a - y - z$	

5. Solve $A = \frac{t}{1-s}$ for s.

Steps	Reasons
$A = \frac{t}{1-s}$	We want s alone:
$A(1-s) = \frac{t}{\cancel{1-s}} \cdot \frac{\cancel{1-s}}{1}$	1. Clear fractions by multiplying both sides by $1-s$.
$A - As = t$ $A - t = As$	2. Get the s term on one side. Notice that miving the terms to the other side changes the sign. We are just adding As to both sides and subtracting t from both sides.
$\frac{A-t}{A} = \frac{\cancel{A}s}{\cancel{A}}$	3. Cancel A by dividing both sides by A.
$s = \frac{A-t}{A}$	That is it.

Exercises

Solve the following literal equations for the given variable.

1. Solve $\mathrm{A} = \mathrm{L} \cdot \mathrm{W}$ for L.
2. Solve $\mathrm{A} = \mathrm{L} \cdot \mathrm{W}$ for W.
3. Solve $\mathrm{P} = 2\mathrm{L} + 2\mathrm{W}$ for L.
4. Solve $5\mathrm{x} + 3\mathrm{y} = 45$ for x.
5. Solve $-3\mathrm{x} + 4\mathrm{y} = -24$ for x
6. Solve $2\mathrm{x} - 3\mathrm{y} = -6$ for y.
7. Solve $4\mathrm{x} - 7\mathrm{y} = 28$ for y.
8. Solve $\mathrm{A} = \mathrm{P} + \mathrm{Pmt}$ for m.
9. Solve $\mathrm{A} = \frac{\mathrm{x}+\mathrm{y}+\mathrm{z}}{3}$ for y.
10. Solve $\mathrm{A} = \frac{\mathrm{x}+\mathrm{y}+\mathrm{z}}{3}$ for z.
11. Solve $\mathrm{M} = \frac{\mathrm{t}}{\mathrm{r}+\mathrm{s}}$ for r.
12. Solve $\mathrm{M} = \frac{\mathrm{t}}{\mathrm{r}+\mathrm{s}}$ for s.
13. Solve $\mathrm{F} = \frac{9}{5}\mathrm{C} + 32$ for C.
14. Solve $\mathrm{A} = \mathrm{P}(1 + \mathrm{rt})$ for r.
15. Solve $\mathrm{A} = \mathrm{P}(1 + \mathrm{rt})$ for t.

3

Here we focus on applied problems or the “dreaded word problem.” An overall method is discussed to give students an approach. Using individual strategies for specific types of applied problems is the key to becoming a good algebraic problem solver. Keep at it and try to determine what “type” of problem is presented.

Course Outcomes:

- Demonstrate mastery of algebraic skills
- Recognize and apply mathematical concepts to real-world situations

3.1 Translation Problems

The arithmetic operations for specific phrases are introduced. The concepts of expression and equation are developed. Students learn to translate phrases into algebraic expressions and algebraic equations by focusing on key words.

3.2 General Strategy for Problem Solving

A general approach to solving application problems is outlined. Students learn to identify specific types of problems and use appropriate methods for them.

3.3 Ratios and Solving Proportions

Rates, ratios, and proportions are defined. Students learn to solve proportions. Students will recognize certain applied problems as proportions.

3.4 Percents

This section begins with a discussion of changing between percents and fractions or decimal numbers. The basic percent equation, mark-up, and mark down are covered. This section may be covered later just prior to Chapter 6.

3.5 Inequalities

Solving inequalities and compound inequalities is discussed. Graphing inequalities in one variable and expressing answers in set-builder notation are explained. Applied problems for inequalities are covered.

Translating words to variable expressions or equations is made easier by knowing the phrases that indicate the various operations and breaking the words down to specific parts. The textbook highlights in blue phrases that indicate the different operations.

Addition

Phrase	Example	Operation
added to	7 is added to 12	7+12
more than	3 more than 8	8+3
the sum of	the sum of 6 and 7	6+7
increased by	9 increased by 8	9+8
the total of	the total of 11 and 15	11+15
plus	2 plus 4	2+4

Subtraction

Phrase	Example	Operation
minus	11 minus 7	11 – 7
less	9 less 5	9 – 5
less than	3 less than 6	6 – 3
the difference between	the difference between 15 and 8	15 – 8
decreased by	20 decreased by 14	20 – 14
subtract…from	subtract 2 from 30	30 – 2

Multiplication

Phrase	Example	Operation
times	12 times 7	12·7
the product of	the product of 11 and 6	11·6
multiplied by	10 multiplied by 3	10·3
twice	twice 5	2·5
a fraction of	$\frac{1}{3}$ of 15	$\frac{1}{3}\cdot 15$
a percent of	20% of 30	0.2·30

Division

Phrase	Example	Operation
the quotient of	the quotient of 40 and 8	40÷8
divided by	12 divided by 3	12÷3

Equality

Phrase	Example	Operation
is	The quotient of 36 and 4 **is** 9.	36 ÷4 = 9
equals	The difference between 30 and 11 **equals** 19.	30 –11 = 19
was	10 minus 4 **was** 6.	10 – 4 = 6
represents	25 increased by 14 **represents** 39.	25 + 14 = 39
is the same as	The product of 7 and 8 **is the same as** 56.	7 · 8 = 56
is equal to	15 more than 20 **is equal to** 35.	20 + 15 = 35

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Power

Phrase	Example	Operation
the square of	the square of y	y^2
the second power of	the second power of y	y^2
the cube of	the cube of t	t^3
the third power of	the third power of t	t^3

Variable expressions do not have an equality sign. (Variable equations have an equality sign.) Do not write an equality sign if the directions ask for a variable expression.

The key to translating words to expressions is breaking down the words into smaller parts:
1. Write the phrase in one long line.
2. Translate words to numbers right away (fifteen is 15)
3. Identify the operations.
4. Often "and" separates two parts. Replace "and" with the appropriate operation.

Examples

1. Translate into a variable expression: “the sum of twice x and fifteen”

Steps	Reasons
the sum of twice x and 15	Write the phrase as one long line. Translate the number.
twice x + 15	Sum indicates addition at the "and"
$2x + 15$	Twice means multiply by 2.

2. Translate into a variable expression and simplify:
“fifteen less than the difference between a number and six”

Steps	Reasons
15 less than the difference between a number and 6	Write the phrase as one long line. Translate the numbers.
15 less than x – 6	Difference indicates subtraction. Replace "and" with subtraction. Replace "a number" with your favorite letter.
$(x - 6) - 15$	"less than" indicates subtraction.
$x - 6 - 15$	The directions also say to simplify.
$x - 21$	Now it is done.

3. Translate into a variable expression and simplify:
“the difference between a number and four added to the sum of the number and eight”

Steps	Reasons
the difference between a number and 4 added to the sum of the number and 8	Write the phrase as one long line.
↓ ↓ ↓	
(the difference between a number and 4) + (the sum of the number and 8)	"added to" is replaced with addition.
$(x - 4) + (x + 8)$	Difference indicates subtraction. Sum indicates addition.
$2x + 4$	Collect like terms.

Sometimes one variable is needed to express two concepts. The second number can be figured out by subtracting. Consider the following sentence and a few random possible values for the first number: “The sum of two numbers is 20.” That would lead to following scenarios.

First number	Second number	Operation
12	8	$20 - 12 = 8$
7	13	$20 - 7 = 13$
18	2	$20 - 2 = 18$
x	?	$20 - x$

That is the key. If the sum of two numbers is 20, let the first number be x and the second number is $20 - x$.

Examples

4. Translate into a variable expression and simplify:

The sum of two numbers is fifteen. Using z to represent the larger number, translate "eighteen more than the smaller number".

Steps	Reasons
“18 more than the smaller number”	We need to translate.
the smaller number is $15 - z$	If z represents the larger number and the sum of two numbers is 15, then the smaller number is $15 - z$ as shown above
18 more than $15 - z$	Replace the smaller number with $15 - z$ More than means add.
$15 - z + 18$	Collect like terms to simplify.
$33 - z$	

5. Translate into a variable expression and simplify:

The sum of two numbers is thirty. Using p to represent the larger number, translate "the difference between twice the smaller number and seven" into a variable expression.

Steps	Reasons
the difference between twice the smaller number and 7	Translate.
the smaller number is $30 – p$	If p represents the larger number and the sum of two numbers is 30, then the smaller number is $30 – p$. See the explanation before this set of examples.
the difference between twice $(30 – p)$ and 7	Replace the smaller number with $30 – p$.
twice $(30 – p) – 7$	Difference indicates subtraction. Replace "and" with subtraction.
$2(30 – p) – 7$	Twice means multiply by 2.
$60 – 2p – 7$	Simplify.
$– 2p + 53$	Collect like terms.

Now that we have been translating into variable expressions, we will translate phrases into variable equations. Variable equations include an equality sign as well as numbers, variables, and arithmetic operations. Translating words into equations allows us to find an unknown value when we solve the equations.

Keys to “translating word” problems:

1. Let a variable stand for what you are asked to find, which will be “the number”.
2. Writing = where indicated from the beginning splits the more complicated sentence into two parts.
3. Translate the two parts of the problem.
4. Solve and state answer.

Examples

6. A number increased by twenty-seven is fifty. Find that number.

Steps	Reasons
x = the number	It is usually a good idea to let the variable stand for what you are looking for.
A number increased by twenty-seven is fifty. ↓ ↓ ↓ ↓ ↓ x + 27 = 50	Write the sentence on a single line. Translate individual words.
$x + 27 = 50$ $\quad -27 \quad -27$ $x = 23$	Solve the equation.

So, the number is 23.

7. Three hundred represents twice a number. Find that number.

Steps	Reasons
x = the number	It is a good idea to let the variable stand for what you are looking for.
Three hundred represents twice a number. ↓ ↓ ↓ ↓ 300 = 2 · x	Write the sentence on a single line. Translate individual words. Look for the "= word", which is "represents".
$300 = 2x$ $\frac{300}{2} = \frac{2x}{2}$ $150 = x \ or \ x = 150$	Solve the equation.

The number is 150.

8. Negative three-eighths is equal to the product of two-thirds and some number. Find the number.
Solution:
Write the phrase and translate the different parts. "is equal to" splits the problem in half. Product means we multiply.

Negative three-eighths equals the product of two-thirds and some number
$-\frac{3}{8}$ = $\frac{2}{3}$ · x

$-\frac{3}{8}=\frac{2}{3}x$	Solve the resulting equation. The fraction is connected to the variable by multiplication. We should divide both sides by $\frac{2}{3}$.
$-\frac{3}{8}\cdot\frac{3}{2}=\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}x$	Dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{3}{2}$.
$-\frac{9}{16}=x$	Do the multiplication.
$x=-\frac{9}{16}$	Write your answer with the variable on the left.
The number is $-\frac{9}{16}$.	State the answer.

9. The difference between a number and negative seven is negative sixteen. Find the number.

Steps	Reasons
x = the number	Let a variable stand for the quantity that you are trying to find.
The difference between a number and –7 is –16. x – –7 = –16	Write out the sentence in a row. Translate the numbers immediately.
$x-\left(-7\right)=-16$ $x+7=-16$	Difference represents subtraction. Subtraction replaces the “and.“ = replaces “is”
$x+7=-16$ $-7 \quad -7$	Solve the equation: Simplify the – (–) as +
$x=-23$ The number is –23.	Because the number is connected to the variable by addition, subtract both sides by 7.

10. The sum of two numbers is thirty-three. Twice the smaller is three more than the larger number. Find the two numbers.

Steps	Reasons
x = the smaller number? **= the larger number?**	I am looking for two numbers. One of them will be my variable. The other one also needs to be named, but that needs to wait. I do not want another variable because we do not know how to solve equations with more than one variable.
Sum of two numbers is 33. **Twice smaller is 3 more than larger.**	List both pieces of information.
Sum of two numbers is 33. **Smaller + larger = 33** **–smaller – smaller** **= 33 – x**	Look at the simpler piece of information. If they add up to 33, then we can solve for either number by subtracting on both sides. It would be great for students to know **that if the sum is 33 then the numbers being added are x and 33 – x.**
Twice smaller is 3 more than larger. **2(x) = (33 – x) + 3**	Now look at the more complicated piece of information. Substitute smaller with x and larger with 33-x. Twice means multiply by 2, 3 more than means add 3, and "is" means =.
$2x = 33 - x + 3$ $+x \qquad +x$ $3x = 36$ $\frac{3x}{3} = \frac{36}{3}$ $x = 12$	Solve the equation
x = 12 **33 –x = 33 – 12 = 21**	x = 12 so find the larger number by evaluating 33 – x for x = 12.
The numbers are 12 and 21.	State your answer.

Check your answers by looking at the original problem.
12 + 21 = 33
2(12) = 3 more than 21.

Exercises

Translate each of the following to an algebraic expression and simplify if possible. Remember that algebraic expressions do not include an equal sign.

1. The sum of five and three times a number
2. The difference between a number and fifty percent of the number
3. The difference between twice a number and 5
4. The sum of seven more than twice a number and the number
5. The difference between negative three times a number and five more than twice the number
6. The sum of twice a number decreased by ten and twenty percent of the number
7. Twenty less than twelve percent of a number minus the sum of twice the number and negative fifteen
8. Thirty more than eighty percent of a number added to the difference of three times the number and negative twenty-five
9. The sum of two numbers is thirty. Using z to represent the larger number, translate "twenty more than the smaller number" into a variable expression and simplify.
10. The sum of two numbers is forty. Using x to represent the larger number, translate "fifty more than the smaller number" into a variable expression and simplify.
11. The sum of two numbers is ten. Using x to represent the smaller number, translate "the difference between twice the larger number and negative twenty" into a variable expression and simplify.
12. The sum of two numbers is twenty-five. Using z to represent the smaller number, translate "the sum of twice the larger number and fifteen" into a variable expression and simplify.

Translate the following into an algebraic equation and solve to find the missing number. Remember that variable equations do have an equal sign.

13. The difference between a number and negative twenty-five is seventy-three. Find the number.

14. Three-fourths of a number minus five-eighths equals one-half. Find the number.

15. The sum of two-thirds of a number and negative one-ninth is five-sixths. Find the number.

16. The sum of two numbers is thirty-four. The sum of twice the smaller number and fifteen is thirty-nine. Find the two numbers.

17. The sum of two numbers is forty. The difference between twice the smaller and forty-two is negative fourteen. Find both numbers.

18. The sum of two numbers is fifteen. The difference between twice the larger number and twenty-two is eight. Find both numbers.

19. The sum of two numbers is negative seven. The sum of twice the smaller number and seventy is six. Find both numbers.

Application problems or word problems are often a source of frustration for many students. Not to worry! We can improve our problem solving skills. Frequently students are overwhelmed by the information and do not know where to begin or they protest that they do not know what is being asked. If there is a question mark, look there! In algebra, we often let a variable stand for what we are being asked. Following the steps below will help us to become better problem solvers.

General Strategy for Solving Application Problems

1. Familiarize yourself with the problem:
    - Read the problem perhaps several times.
    - Identify the question. Frequently, let a variable stand for what is being asked.
    - List information, write down important formulas, pictures
    - If you can use some information to let one variable stand for two unknowns, do it.
    - Charts: Many problems can be organized by using charts. There are specific types of charts for different types of problems.
2. Translate to a solvable problem:
    - We translate word problems into equations that can be solved.
3. Solve the equation.
    - Usually the equations are not too complicated.
4. Check:
    - Does the answer make sense? Remember, bicycles to do not go 300 mph, planes do not fly 4 miles in 7 hours, butterflies do not weigh 750 kilograms, etc.
5. State answer:
    - Use a sentence. Most word problems have a context and it is important to state whether we are talking about area or length or speed or time.
    - Use the correct units. Depending on the problem, the correct answer may use square feet, meters, miles per hour or seconds.
6. Look Back:
    - To learn how to do word problems it is important to apply the correct methods to different types of problems. After you struggle to finish a word problem you need to take some time to focus on what you did well to solve the problem.
    - Ask yourself two questions:
        1. What type of problem is being solved?
        2. What method is employed to solve the problem? Especially pay attention to the types of charts that are being used.

Do look back! It is extremely helpful when learning how to solve word problems.

Examples

1. Find three consecutive odd integers such that twice the difference of the third and first integers is fifteen less than the second integer.

Steps	Reasons
x = first odd integer x + 2 = second odd integer x + 4 = third odd integer	Use 1 variable to name all three integers.

twice the difference of the third and first integers is 15 less than the second integer

2 · (difference of the third and first integers) = second integer – 15

2 · [(x + 4) – x] = x + 2 – 15

$2[(x+4)-x]=(x+2)-15$	Solve for x.
$2[x+4-x]=x+2-15$ $2(4)=x-13$	The parentheses can be dropped because there is nothing in front of the parentheses to distribute. Collect like terms.
$8=x-13$	
$21=x$	Get x by itself by adding 13 to both sides.
$x=21$ $x+2=21+2=23$ $x+4=21+4=25$	Find the other odd integers by adding 2 and 4.
The three integers are 21, 23, and 25.	State answer.

2. A stamp collector bought 250 stamps for $61.50. The purchase included 10¢ stamps, 25¢ stamps, and 30¢ stamps. The number of 25¢ stamps is five times the number of 10¢ stamps. How many 30¢ stamps were purchased?

number of 25¢ stamps is 5 times the number of 10¢ stamps.

number of 25¢ stamps = 5 · number of 10¢ stamps
number of 25¢ stamps = 5 · x
number of 10¢ stamps = x

We can use a chart to get the appropriate equation.

	Number of stamps	Cost of each stamp	Total cost
10¢ Stamps	x	.10	.10x +
25¢ Stamps	5x	.25	.25(5x) +
30¢ Stamps	250 – 6x ???	.30	.3(250 – 6x) =
Totals	250		61.50

Notice: We are looking for the number of 30¢ stamps, which is equal to the total number of stamps minus the number of 10¢ number and minus the number of 25¢ stamps. Add the cost for the 10¢, 25¢, and 30¢ stamps to get the cost of the collection.

Solve the equation from the Total cost column of the chart.

$.10x + .25(5x) + .3(250 - 6x) = 61.5$	Solve the equation.
	Distribute the .3
$.10x + 1.25x + 75 - 1.8x = 61.5$	Collect like terms: .10 + 1.25 – 1.8 = –.45
$-.45x + 75 = 61.5$	Subtract 132 from both sides.
$-.45x = -13.5$	Divide both sides by –.45
$x = \dfrac{-13.5}{-.45}$	
$x = 30$	

x = 30
250 – 6x = 250 – 6·(30) = 250 – 180 = 70 Get the number 30¢ stamps

The stamp collector bought seventy 30¢ stamps. State the answer.

3. An IQ training program claims that for every hour of training an individual will improve their IQ by 1.5. If we accept that claim as fact, how many hours of training will it take for somebody with an IQ of 84 to raise their IQ score to 150?

Let x stand for what we are trying to find:
x = the number of hours of training??? (the question)

Think of the process:

New IQ equals the old IQ plus increase from training
New IQ = 84 + 1.5 times the number of hours if training

150 = 84 + 1.5 x

Solve the equation:

$150 = 84 + 1.5\mathrm{x}$

$150 - 84 = 1.5\mathrm{x}$

$66 = 1.5\mathrm{x}$

$\frac{66}{1.5} = \mathrm{x}$

$\mathrm{x} = 44$

If the claim were to be true after 44 hours of training, somebody could improve their IQ from 84 to 150.

4. The average weight for three year old girls is 30 pounds. If each year the average female child gains 7.2 pounds, how old will the average girl be when she weighs 102 pounds?

List information:
3 years old ⟶ 30 pounds
Each year after 3 years old ⟶ 7.2 pounds

Since 3 years old is the starting point and we are looking for the age or number of years old we will let:
x = the number of years after 3 years old

End weight = starting weight + weight added each year
102 = 30 + 7.2 times number of years after 3 (which is x)

$102 = 30 + 7.2x$ Write the equation.

$102 - 30 = 7.2x$

$72 = 7.2x$ Solve the equation.

$\frac{72}{7.2} = x$

$x = 10$ The average girl will weigh 102 pounds when she is 13 years old.

$10 + 3 = 13$

5. The local swimming pool has two summer plans to choose from. For the first plan an individual pays $100 for the whole summer with unlimited use. For the second plan individuals pay $50 for the summer and then $2.50 per day that they use the pool. After how many days will the costs of the two plans be the same?

When comparing two different situations a chart with a line down the middle helps organize the information.

First plan	Second plan
$100 for the summer	$50 for the summer $2.50 per day used

Total cost
100

Same
Cost

Total cost
50 + 2.50 x (number of days using the pool)

x= the number of days using the pool?

$100 = 50 + 2.50\text{x}$ Write the equation.
$100 - 50 = 2.50x$
$50 = 2.50x$ Solve the equation.
$\frac{50}{2.50} = x$
$20 = x$

After 20 days of using the pool the cost of both plans is the same.

6. Janet wants to know which cell phone plan will suit her best. The first plan costs $35 per month and $0.10 for each additional minute after the first 90 minutes. The second plan costs $30 per month with a cost of $0.15 after the first 90 minutes. After how many minutes will the cost of the two plans be the same?

When comparing two different situations a chart with a line down the middle helps organize the information.

First plan	Second plan
$35 per month	$30 per month
$0.10 per minute after 90 minutes	$0.15 per minute after 90 minutes

Total Cost
35 + 0.10 x (number of minutes more than 90)

Same
Cost

Total Cost
30 + 0.15 x (number of minutes more than 90)

x= the number of minutes after the first 90 minutes??
Use the equality sign for the same cost.

$35 + 0.10\text{x} = 30 + 0.15\text{x}$ Write the equation.
$35 - 30 = 0.15\text{x} - 0.10\text{x}$ Solve.
$5 = 0.05\text{x}$
$\frac{5}{.05} = \text{x}$
$\text{x} = 100$

After 100 + 90 or 190 minutes of calls the cost of the two cell phone plans will be the same.

Looking back we really have a few different types of problems. Example 1 involves consecutive odd integers. Odd integers are two apart (1,3,5,7,…). So, we can label the unknown integers x, x+2, x+4. Consecutive even integers (2,4,6,8,…) would work the same way since they are also two apart. Consecutive integers (1,2,3,4,…) are one apart. If a problems asks about consecutive integers, they can be labeled x, x+1, x+2.

Example 2 is a stamp problem. It is much like a translating problem to get the different names of the unknown quantities of stamps. Multiplying the number of stamps by the value of the stamps lets us get the total value for each type of stamp, which can be added together to get the total value of the collection.

Examples 3 and 4 start at a given level and then increase for each hour of training or year of growth. So, the end value is equal to the beginning values plus the hourly or yearly increase times the number of hours or years.

In examples 5 and 6, we are comparing two situations. When comparing two situations, it is useful to make a chart to organize the different information. The two types of information are connected in some way. In the above examples, we are looking for the condition that will make the two costs the same.

Exercises

To solve the following word problems use algebraic equations. State your answer with the correct units if appropriate.

1. Find three consecutive odd integers such that the three times the difference of the third and first odd integer is five less than the second odd integer.

2. Find three consecutive even integers such that twice the sum of the first and third even integers is forty-four more than twice the second even integer.

3. Find three consecutive integers such that five times the third integer is eleven less than three times the sum of the first and the second integers.

4. Find three consecutive integers such that six times the first integer is sixteen more than the twice the sum of the second and third integers.

5. A stamp collector bought 160 stamps for $25.00. The purchase included 5¢ stamps, 15¢ stamps, and 25¢ stamps. The number of 15¢ stamps is three times the number of 5¢ stamps. How many of each type of stamp was purchased?

6. A stamp collector bought 295 stamps for $87.75. The purchase included 5¢ stamps, 25¢ stamps, and 40¢ stamps. The number of 40¢ stamps is fifty more than twice the number of 5¢ stamps. How many of each type of stamp was purchased?

7. If the average salary for a college instructor in 1925 was $2300 and that average salary increased each year by $180, in what year was the average salary for a college instructor $10,040?

8. If the average salary for a young businessperson in 1943 was $4530 and that salary increased by $370 each year, in what year was the average salary for a young businessperson $17,110?

9. A printer that costs $60 requires replacement cartridges that sell for $23. Over the life of the printer the cost of the printer and replacement cartridges is $865. How many cartridges were used over the life of the printer?

10. The air pressure at sea level is about 14.7 pounds per square inch (psi). For every foot above sea level, the air pressure drops about 0.0005263 psi. According to this information, at about what height above sea level is the air pressure 13.2 psi?

11. The Scholastic Aptitude Test (SAT) is used by universities to help determine if a candidate should be allowed to enter the school. One year at Stanford 54% of the students that were admitted had a SAT Math score of 700 or better. An SAT training program claims that for every hour of training an individual will improve their SAT Math score by 3 points. If we accept that claim as fact, how many hours of training will it take for somebody with a SAT Math score of 529 to raise their SAT Math score to 700?

12. The average weight for four year old boys is 34.5 pounds. If each year the average boy gains 8.2 pounds, how old will the average boy be when he weighs 124.7 pounds?

13. A day care offers two options. Plan 1 allows a child to attend the day care any time during the work week for a cost of $275 per month. Plan 2 requires a $50 per month fee plus $3.75 per hour. After how many hours will the costs of the two plans be the same?

14. An electric company will allow its clients to pick one of two different options. For the first option clients pay a $37.24 basic services fee plus $0.096 per kilowatt hour. The second option allows clients to pay a $24.36 basic services fee with a cost of $0.124 per kilowatt hour. After how many kilowatt hours will the costs of the two plans be the same?

15. An electric company will allow its clients to pick one of two different options. For the first option clients pay a $27.25 basic services fee plus $0.162 per kilowatt hour. The second option allows clients to pay a $43.49 basic services fee with a cost of $0.104 per kilowatt hour. After how many kilowatt hours will the costs of the two plans be the same?

16. Rental car company A charges $125 per week with a $0.04 charge per mile and rental car company B charges $59 per week with a $0.10 charge per mile. After how many miles with the cost of the two rental car companies be the same?

Ratios are quotients or quantities with the same units. The quotient is the result of division. Division can be written as a fraction.

Words	Symbols	Meaning
2 cups of sugar to 8 cups of milk	$2:8 \ or \ 2 \div 8 \ or \ \frac{2}{8}$	$\frac{2 \ cups\,of \ sugar}{8 \ cups\,of \ milk}$

These three ways of expressing a ratio all have the same meaning.

Simplify ratios by writing them as fractions.

Example

1. Simplify: 2 cups of sugar to 8 cups of milk

Steps	Reasons
2 cups of sugar to 8 cups of milk	Translate to fraction notation.
$\frac{2}{8} = \frac{1}{4}$	Simplify the fraction.
$\frac{1}{4} \ or \ 1:4 \ or \ 1 \ to \ 4$	You can write your answer three ways.

Rates are quotients of quantities with different units. The quotient is the result of division. Division can be written as a fraction.

Words	Symbols	Meaning
192 dollars to 150 euros	192 dollars : 150 euros	$\frac{192 \ dollars}{150 \ euros}$

Simplify rates by writing them as fractions.

Example

1. Simplify: 192 dollars to 150 euro

Steps	Reasons
192 dollars to 150 euro	Translate to fraction notation.
$\frac{192}{150} = \frac{6 \cdot 32}{6 \cdot 25} = \frac{32}{25}$	Simplify the fraction.
$\frac{32 \ dollars}{25 \ euros} \ or \ 32 \ dollars : 25 \ euros \ or \ \$32 \ to \ 25 \ euros$	You can write your answer three ways.

A unit rate has a 1 in the denominator. To get a unit rate just divide with a calculator. Unit rates can be used to convert units.

Example

2. Write 192 dollars to 150 euro as a unit rate.

Steps	Reasons
192 dollars to 150 euro	Translate to fraction notation.
$\frac{192}{150} = 1.28$	Divide.
\$1.28 / 1 euro	Write the answer with units.

\$1.28 / euro also can be written \$1.28 : 1 euro. We can use this unit fraction when making conversions euro to dollar.

Proportions are an equality of ratios, rates, or fractions.

To check if a proportion is true or to solve for a variable we can "cross multiply."

We know $\frac{2}{3} = \frac{6}{9}$.

We can check by "cross multiplying" $\frac{2}{3} \times \frac{6}{9}$ $\quad 2 \cdot 9 = 18$ and $3 \cdot 6 = 18$

Since the fractions are equal, we do get the same number when we "cross multiply."

We know $\frac{1}{2} \neq \frac{3}{4}$

We can check by "cross multiplying" $\frac{1}{2} \times \frac{3}{4}$ $\quad 1 \cdot 4 = 4$ and $2 \cdot 3 = 6$

Since the fractions are not equal, we do not get the same number when we "cross multiply."

To solve proportions:
1. "cross multiply"
2. Solve the resulting equations.

Examples

3. Solve: $\frac{x}{25}=\frac{7}{5}$

Steps	Reasons
$\frac{x}{25}=\frac{7}{5}$	We want to solve a proportion.
$\frac{x}{25} \times \frac{7}{5}$	Cross Multiply
$5x=25\times 7$	
$5x=175$	Solve the equation.
$\frac{5x}{5}=\frac{175}{5}$	
$x=35$	

Check the equation:

$\frac{35}{25}=\frac{7}{5}$ is true. When $\frac{35}{25}$ is simplified, it is $\frac{7}{5}$.

4. Solve: $\frac{3}{8}=\frac{9}{x+3}$

Steps	Reasons
$\frac{3}{8}=\frac{9}{x+3}$	We want to solve a proportion.
$\frac{3}{8} \times \frac{9}{x+3}$	Cross Multiply Use parentheses to multiply 3 and x+3
$3(x+3)=8\cdot 9$	Solve the equation:
$3x+9=72$ $-9 \quad -9$	Short-cut: $3x+9=72$ Notice that we can subtract 9 from both sides which gets rid of the 9 on the left.
$3x=63$	$3x=63$ Dividing by 3 on both sides gets rid of the 3 on the left.
$\frac{3x}{3}=\frac{63}{3}$	$x=21$ We save time using this short-cut method.
$x=21$	

5. A car is driven 162 miles and uses 7.5 gallons of gasoline. At this rate how far will the car go with 2.5 gallons of gasoline?

Steps	Reasons
162 miles : 7.5 gallons x = miles?? : 2.5 gallons	Write both rates (or ratios). Keep the same units in the same column. Let a variable stand for the missing piece of information.
$\frac{162}{7.5}=\frac{x}{2.5}$	Write the proportion: 162 miles is to 7.5 gallons as what number of miles is to 2.5 gallons.
$\frac{162}{7.5} \times \frac{x}{2.5}$	Cross Multiply
$7.5x=162\cdot 2.5$	
$7.5x=162\cdot 2.5$	Solve the equation.
$7.5x=405$	
$\frac{7.5x}{7.5}=\frac{405}{7.5}$	
$x=54$	
The car would go 54 miles with 2.5 gallons of gas.	Write a sentence to state the answer with the correct units.

6. In a wilderness area, 8 bears are caught, tagged, and released. Later 20 bears are caught and 5 have tags. Estimate the number of bears in the wilderness area.

Steps	Reasons
8 bear tagged : x = all bear in area 5 tagged : 20 captured	Try to set up two ratios:
So, we get the proportion $\frac{8}{x}=\frac{5}{20}$	The trick is to see that there are a total of 8 bears tagged in the whole area.
$\frac{8}{x} \times \frac{5}{20}$	Cross multiply to solve proportions.
$8\cdot 20=5x$ $160=5x$ $x=32$	There are approximately 32 bears in the wilderness area

7. After working for a company for 15 months, an employee has earned 20 vacation days. At this rate, how much longer will the employee have to work in order to get a 6-week vacation (30 work days)?

Steps	Reasons
15 months : 20 vacat. days 15 +x months :30 vacat. days	Write the proportion as months to vacation days. Let x=number of months in addition to the 15
$\frac{15}{20} = \frac{15+x}{30}$	Write the proportion and cross-multiply to solve.
$15 \cdot 30 = 20(15+x)$	Solve the resulting equation by distributing the 20.
$450 = 300 + 20x$	
$150 = 20x$	
$\frac{150}{20} = x$	
$x = 7.5$	

The employee needs to work 7.5 months more to get the 6-week vacation.

8. An investment of \$3500 earns \$700 each year. At the same rate, how much additional money must be invested to earn \$1200 each year?

Steps	Reasons
investment \$3500 : earns \$700 x + 3500 : earns \$1200 x=additional amount to invest??	Write both rates (or ratios). Keep the same type of information in the same column. Let a variable (x) stand for the missing piece of information. Here we want the additional amount more than the \$3500 to invest.
$\frac{3500}{700} = \frac{x+3500}{1200}$	Write the proportion.
$\frac{3500}{700} \times \frac{x+3500}{1200}$	Cross-multiply.

$$700(x+3500)=3500\cdot 1200$$

$$\begin{array}{r} 700x+2{,}450{,}000=4{,}200{,}000 \\ -2{,}450{,}000 \quad -2{,}450{,}000 \end{array}$$

$$700x=1{,}750{,}000$$

$$\frac{700x}{700}=\frac{1{,}750{,}000}{700}$$

$$x=2500$$

Solve the equation.

Short-cut:

$$700x+2{,}450{,}000=4{,}200{,}000$$
$$700x=1{,}750{,}000$$
$$x=2500$$

Subtract 2,450,00
from both sides.
Divide both sides
by 700.

The above short-cut is quicker than what we did on the left. Either way is fine.

$2500 more must be invested to earn $1200 each year.

Exercises

Express the following as unit rates:

1. 220 EUR (European Euro) is paid for an item costing 275 USD (US Dollars)

2. 56.70 USD (US Dollars) is paid for an item costing 35 GBP (Great Britain Pound)

3. 183.52 CAD (Canadian dollar) is paid for an item costing 148 USD (US Dollars)

Solve the following proportions:

4. $\frac{10}{6} = \frac{25}{x}$

5. $\frac{6}{21} = \frac{x}{56}$

6. $\frac{36}{x} = \frac{12}{3}$

7. $\frac{x}{72} = \frac{5}{12}$

8. $\frac{x+2}{12} = \frac{15}{18}$

9. $\frac{6}{x-3} = \frac{8}{12}$

10. $\frac{14}{3} = \frac{35}{x-4}$

11. $\frac{x-5}{6} = \frac{x+4}{9}$

12. $\frac{9}{x-3} = \frac{15}{x+3}$

13. A car is driven 127.4 miles and uses 5.2 gallons of gasoline. At this rate how far will the car go with 3.6 gallons of gasoline?

14. A car is driven 317.1 km and uses 30.2 liters of gasoline. At this rate how far will the car go with 40.5 liters of gasoline?

15. In order to estimate the number of elephants in an enclosed habitat, 28 elephants are caught, tagged, and released. Later 105 elephants are observed and 12 have the tags. Estimate the number of elephants in the enclosed habitat.

16. In a nature preserve 126 elk are caught and tagged. Later 75 elk are caught in the same preserve and 45 are found to have tags. Estimate the number of elk in the nature preserve.

17. In a wildlife refuge 42 bears are caught and tagged. Later 21 bears are caught in the same wildlife refuge and 14 are found to have tags. Estimate the number of bears in the wildlife refuge?

18. After working for a company for 12 months, an employee has earned 15 vacation days. At this rate, how much longer will the employee have to work in order to get a 4-week vacation (20 work days)?

19. After working for a company for 18 months, an employee accrued 12 vacation days. At this rate, how much longer will the employee have to work in order to get a 5-week vacation (25 work days)?

20. An investment of $6200 earns $800 each year. At the same rate, how much additional money must be invested to earn $1500 each year?

21. An investment of $4500 earns $300 each year. At the same rate, how much additional money must be invested to earn $450 each year?

Percent means per 100. We can replace the percent symbol (%) with multiplication by $\frac{1}{100}$ or 0.01.

Examples

(Changing percents to decimals.)

1. Write 37.5% as a decimal.

Steps	Reasons
37.5%	
	Replace % with multiplication by 0.01.
$37.5 \cdot 0.01$	
	Multiplying by 0.01 moves the decimal point to places to the left.
$.375$	

(Changing percents to fractions.)

2. Write 25% as a fraction.

Steps	Reasons
$25\% = 25 \cdot \frac{1}{100}$	Replace % with $\frac{1}{100}$.
$\frac{25}{1} \cdot \frac{1}{100} = \frac{25}{100}$ $= \frac{1}{4}$	Write the whole number as an improper fraction, multiply, and simplify the fractions.

3. Write $15\frac{2}{5}\%$ as a fraction.

Steps	Reasons
$15\frac{2}{5} \cdot \frac{1}{100}$	Replace % with $\frac{1}{100}$.
$\frac{77}{5} \cdot \frac{1}{100} = \frac{77}{500}$	Write the mixed number as an improper fraction and multiply the fractions.

100% is equal to one. When changing decimal numbers or fractions to percents, we can just multiply by 100% because we are multiplying by one.

(Changing fractions to percents.)

4. Write $\frac{9}{20}$ as a percent.

Steps	Reasons
$\frac{9}{20}$	Multiply by 100%, which is 1.
$\frac{9}{20} \cdot 100\%$	
$\frac{9}{20} \cdot \frac{100}{1}\%$	Write 100 as an improper fraction, multiply, and simplify the fractions.
$\frac{9}{20} \cdot \frac{5 \cdot 20}{1}\%$	
$\frac{45}{1}\%$	
45%	

5. Write $\frac{5}{6}$ as a percent.

Steps	Reasons
$\frac{5}{6} \cdot 100\%$	Multiply by 100%.
$\frac{5}{6} \cdot \frac{100}{1}\%$	Write 100 as an improper fraction, multiply, and simplify the fractions. Write your answer using a mixed number if necessary.
$\frac{5}{2 \cdot 3} \cdot \frac{2 \cdot 50}{1}\%$	
$\frac{250}{3}\%$	
$83\tfrac{1}{3}\%$	

(Changing decimals to percents.)

6. Write 0.64 as a percent.

Steps	Reasons
0.64	
	Multiply by 100%.
$0.64 \cdot 100\%$	Multiplying by 100% moves the decimal point to places to the right.
64%	

7. Write 14.2 as a percent.

Steps	Reasons
14.2	
	Multiply by 100%.
$14.2 \cdot 100\%$	Multiplying by 100% moves the decimal point to places to the right.
1420%	

It may seem strange to get 1420%, but it makes more sense if we keep in mind that 1=100%. So a number more than 1 has a value of more than 100%.

The basic percent equation states that the Amount is the Percent of the Base or A = P · B. It may be easier to translate directly using = for “is” and multiply by percent for “percent of”.

Examples

8. 35% of 80 is 28.

$$(.35)\cdot\ 80 = 28$$

Translate "is" to "=" .
Translate "35%" to .35
Translate "of" to multiplication
Translate the other numbers directly.

One way to solve these percent problems is to translate directly to an equation.

Examples

(Basic percent equation.)

9. 20 is 40% of what? Use the basic percent equation.

Steps	Reasons
20 is 40% of what?	Translate "is" to "=" .
20 = .40 · x	Let a variable stand for what you are looking for. Translate "of" to multiplication Change 40% to .40 and write the numbers.
$20 = .4x$	Solve the equation.
$\frac{20}{.4} = \frac{.4x}{.4}$	
$50 = x$	

20 is 40% of 50.

10. 120 is what percent of 800? Use the basic percent equation.

Steps	Reasons
120 is what percent of 800?	Translate "is" to "=" .
120 = P · 800	Let a variable stand for what you are looking for. Translate "of" to multiplication
$120 = P \cdot 800$	Solve the equation.
$\frac{120}{800} = \frac{P \cdot 800}{800}$	
$.15 = P$	
$P = 15\%$	Change the decimal number to a percent. $.15 = .15 \cdot 100\% = 15\%$

120 is 15% of 800.

To solve percent increase and percent decrease problems we use three pieces of information.

1. **Base:** first value before the increase or decrease. We could think of this as the original amount before the increase or decrease.

2. **Amount:** amount of the increase or decrease;
   - final value after increase minus the first value before the increase
   - first value before decrease minus final value after the decrease

3. **Percent:** percent increase or percent decrease

Examples (Percent increases)

11. There were 8.2 million people in a large city in the year 1990. In the year 2000, there were 10.1 million people living there. Find the percent increase in the city's population from 1990 to 2000.

Steps	Reasons
Base: 8.2 million Amount: 10.1-8.2 = 1.9 mill. Percent: ???	1. **Base:** first value before the increase (original amount) 2. **Amount:** amount of the increase; final value after increase minus first value before increase 3. **Percent:** percent increase is what we are asked to find.
$P \cdot B = A$	Here we are saying that the percent of the increase times the original amount is equal to the amount of the increase.
$P \cdot (8.2) = 1.9$	
$\frac{P \cdot (8.2)}{8.2} = \frac{1.9}{8.2}$	
$P = .2317 \ or \ 23.2\%$	Change the decimal to a percent.
There was a 23.2% increase in the city's population from 1990 to 2000.	State your answer.

Example (Percent decreases)

12. It is estimated that the value of a new car drops 45% in the first two years. Find the value of a $24,900 new car after two years.

Steps	Reasons
Base: 24,900 Amount: 24,900 – unknown = ???? Percent: 45%	1. **Base:** first value before the decrease 2. **Amount:** amount of the decrease; first value before decrease minus final value after decrease. Here we are looking for the final value after the decrease. 3. **Percent:** percent decrease
$P \cdot B = A$	Here we are saying that the percent of the decrease times the original amount is equal to the amount of the decrease.
$.45 \cdot 24{,}900 = A$	
$11{,}205 = A$	We get the amount of the decrease.
original value – decrease = final value 24,900 – 11,205 = 13,695	We subtract the amount decreased from the new car price to get the price after two years.
The value of the car after two years is $13,695.	State the answer using a sentence and correct units.

There are a lot of equations that can be used to do mark-up (increases) and discount (decreases) equations. I will focus on the process, which is familiar to us. **Mark-up** is what a store does to make a profit. Sometimes stores sell their items at a **discount** so that they can sell them quickly.

Mark-up:

Cost of the item + Mark-up = Selling price

Mark-up = a percent of the cost.
(percent · cost)

It should make sense that a company has its cost for an item and then the company charges more in order to make a profit.

Examples: (Mark-up)

13. A shirt with a cost of $15 is sold for $26.70. Find the mark-up rate.

Steps	Reasons
Cost of the item + Mark-up = Selling price	• Write the basic concept behind mark-up rate. • Find the mark-up
15 + Mark-up = 26.7 –15 –15	
Mark-up = 11.7	• Mark-up = a percent of the cost. • "Percent of" means multiply by a percent. So,
Mark-up = P · cost 11.70 = P (15)	• Mark-up = P · cost
$11.7 = P(15)$	Solve the equation.
$\frac{11.7}{15} = \frac{P(15)}{15}$	
$P = .78 \;\; or \;\; 78\%$	There is a 78% mark-up.

14. A mark-up rate of 42% is applied to a tennis racquet costing $75. Find the selling price.

Steps	Reasons
Cost of the item + Mark-up = Selling price	• Write the basic concept behind mark-up.
75 + Mark-up = x	• Let x = selling price, because we are asked to find selling price.
75 + .42(75) = x	• Mark-up = a percent of the cost. • Mark-up = 42% of 75 or .42(75)
$75 + .42(75) = x$ $75 + 31.5 = x$ $106.5 = x$	Solve the equation. The selling price is $106.50

Discounts:

Sale price = Regular price – Discount

Discount = percent of the regular price
(percent · regular price)

This way of thinking should make some sense. When an item is 20% off, we take a discount off the regular price and we say that discount is 20% of the regular price. It works just like a decrease problem where we take a percent of the original amount, which is called the regular price for an item on sale.

Examples:

15. A home entertainment system that has a regular price of $475 is on sale for 25% off the regular price. What is the sale price for the home entertainment system?

Steps	Reasons
Sale price = Regular price – Discount	Write the basic concept behind discount.
x = 475 – .25(475)	Discount is a percent of regular price. (25% of 475 = .25 · 475) Let x = sale price, because we are asked to find sale price.
$x = 475 - .25(475)$ $x = 475 - 118.75$ $x = 356.25$	Solve the equation.
The sale price of the home entertainment system is $356.25	State your answer as a short sentence with the correct units.

16. A blender with a sale price of $48.75 is on sale for 35% off the regular price. Find the regular price.

Steps	Reasons
Sale price = Regular price – Discount 48.75 = x – .35 x	• Write the basic concept of a discount. • Let x = regular price, because we are asked to find regular price. • Discount is a percent of regular price. (35% of unknown = .35 · x). The trick is that we are taking 35% of what we are trying to find which is the regular price.
$48.75 = x - .35x$	Solve the equation.
$48.75 = 1x - .35x$	Think of x as 1x. Then subtract 1 – .35 to collect like terms.
$48.75 = .65x$	
$\frac{48.75}{.65} = \frac{.65x}{.65}$	
$75 = x$	
The blender’s regular price is $75.	State your answer.

Exercises

Change the percent to a decimal number:

1. 12%

2. 120%

3. 0.35%

4. 11.3%

Change the percent to a fraction:

5. 45%

6. $12\frac{3}{4}\%$

7. $47\frac{2}{3}\%$

8. 158%

Change the fraction to a percent:

9. $\frac{7}{25}$

10. $\frac{11}{40}$

11. $\frac{5}{9}$

12. $\frac{2}{15}$

13. $3\frac{1}{3}$

14. $2\frac{5}{12}$

Change the decimals to a percent:

15. 0.73

16. 0.148

17. 0.1682

18. 0.9453

19. 12.29

20. 125.42

Solve the following using the basic percent equation or by translating:

21. What is 27% of 160?

22. 42 is 35% of what?

23. 15.3 is 85% of what?

24. What is 60% of 120?

25. 312 is what percent of 600?

26. 54 is what percent of 45?

27. What is 240% of 80?

28. 72 is 150% of what?

29. 51 is 68% of what?

30. What is 320% of 5?

31. 153.6 is what percent of 240?

32. 486 is what percent of 300?

Solve the following:

33. Student enrollments at a university increased from 8900 to 10,502. Find the percent increase of the student enrollments at the university.

34. The number of city employees increased from 12,250 to 16,170 under a new mayor. What is the percent increase of city employees under the new mayor?

35. The number of jobs in a certain state decreased from 6.0 million to 5.1 million because of an economic crisis. What is the percent decrease in the number of jobs in the state because of the economic crisis?

36. Over his lifetime an adult male's height went from 73 inches to 71 inches. Find the percent decrease in height of the adult male to the nearest hundredth of a percent.

37. A blender that usually sells for $75 is on sale for 35% off of the original amount. What is the sale price of the blender?

38. A laptop that is normally $575 is sold at a discount of 15% off of the regular price. What is the sale price of the laptop?

39. A coffeemaker with a sale price of $95.04 is 28% off of the regular price. What is the regular price of the coffee maker?

40. A new car with a sale price of $22,360.80 is 16% off of the regular price. What is the regular price of the new car?

41. It is estimated that the value of a new car drops 24% in the first year. Find the value of a $32,900 new car after the first year.

42. A retailer buys a television for $380 and then marks up the price 45%. What is the retailers selling price for the television?

43. A wine shop buys a French Chardonnay for $6.50 and then marks up the price 80%. What is the wine shop's sale price for the French Chardonnay?

44. A brand name shirt has a cost of $24 for a small store. The store sells the shirt for $43.20 Find the mark-up rate.

45. A supermarket buys shrimp at a cost of $3.80 per pound. The supermarket sells the shrimp for $6.27 per pound. Find the mark-up rate.

Sets are denoted with braces { } and can be written two ways:

1. Roster Method: Lists the elements of the set.
2. Set-builder notation: we describe the set because it is too big to list all the elements.

Roster Method

$A = \{1,2,3,4,5\}$ The set A has elements 1,2,3,4,5

$B = \{1,3,5,7,9\}$ The set B has elements 1,3,5,7,9.

The union of two sets is the set of all elements in either set or both sets. Union is denoted by $\cup$.

Find $A \cup B$ using A and B from above.

$$A \cup B = \{1,2,3,4,5,7,9\}$$

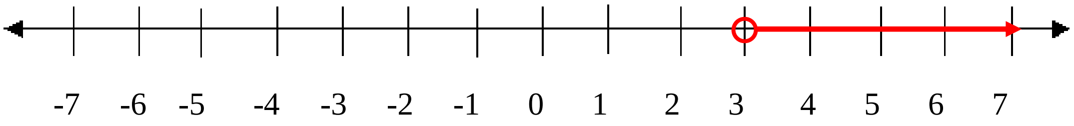
The intersection of two sets is the set of only the elements that are in both sets. Intersection is denoted by $\cap$.

Find $A \cap B$ using A and B from above.

$$A \cap B = \{1,3,5\}$$

Some sets are too big to list the elements. But we can graph them or write them in set-builder notation.

Consider the graph of x > 3



We use an open circle to show that the value 3 is not included.

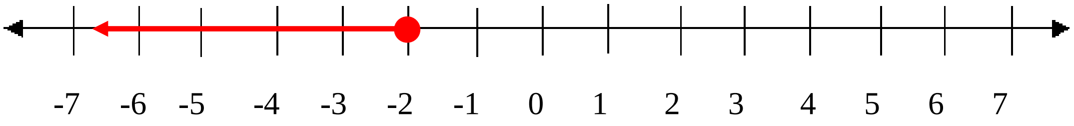
In set-builder notation we write

$$\{x \mid x > 3\}$$

Translated as: "the set of x such that x is greater than 3"

The braces indicate it is a set and the vertical line means "such that".

Consider the graph of $x \leq -2$



We use a closed circle to show that the value -2 is included.

In set-builder notation we write

$$\{x \mid x \leq -2\}$$

Translated as: "the set of x such that x is less than or equal to -2"

The braces indicate it is a set and the vertical line means "such that".

The difference between solving equations and solving inequalities is that we switch the inequality when we multiply or divide both sides by a negative number.

Think of the following example with numbers:

$-3 < 4$ We know that –3 is less than 4.

$-3(-2) > 4(-2)$ Multiply be a negative number switches the sign.

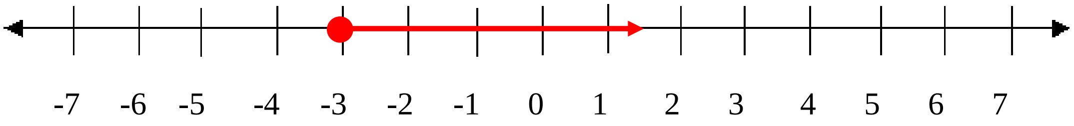
$6 > -8$ The result makes sense.

Examples:

1. Solve: $-3x \le 9$

Steps	Reasons
$-3x \le 9$	
$\frac{-3x}{-3} \ge \frac{9}{-3}$	Dividing both sides by a negative number switches the inequality.
$x \ge -3$	
$\{x \mid x \ge -3\}$	The answer can be written in set-builder notation.

The graph of $x \ge -3$

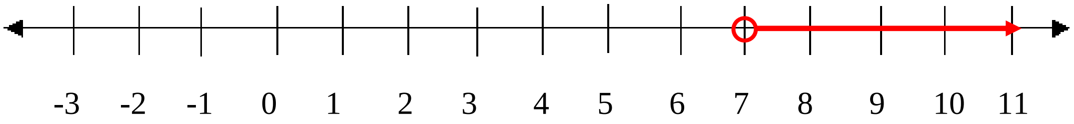


As with equations, we can solve more complicated inequalities by getting all the variables on one side and the numbers on the other. We may need to first simplify both sides of the inequalities.

2. Solve: $5x-4>4x+3$

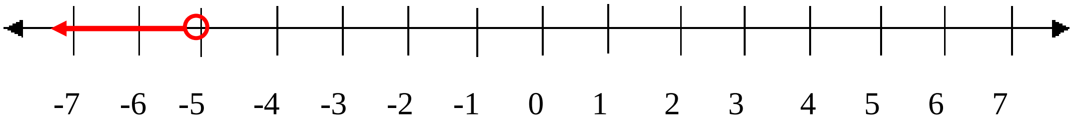
Steps	Reasons
$5x-4>4x+3$	
$-4x \quad -4x$	Subtracting the same number from both sides does not affect the inequality.
$x-4 > 3$ $+4 \quad +4$	Adding the same number to both sides does not affect the inequality.
$x>7$	All done.
$\{x \mid x>7\}$	The answer can be written in set-builder notation.

The graph of x > 7. There is an open circle at 7 because 7 is not included for >.



3. Solve: $2x-11>5x+4$

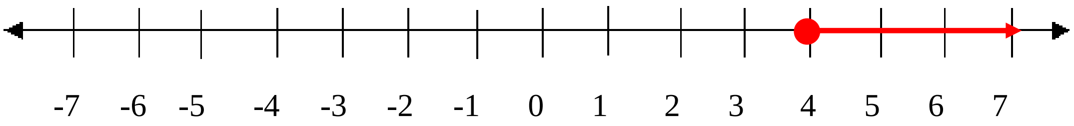
Steps	Reasons
$2x-11>5x+4$	
$2x-5x>4+11$	Add 11 and subtract 5x from both sides of the equation.
$-3x>15$	
$\frac{-3x}{-3}<\frac{15}{-3}$	Dividing both sides by a negative number switches the inequality.
$x<-5$	
$\{x \mid x<-5\}$	We can write the answer using set-builder notation.



4. Solve: $3(2x-9)+2\geq 17-2(x+5)$

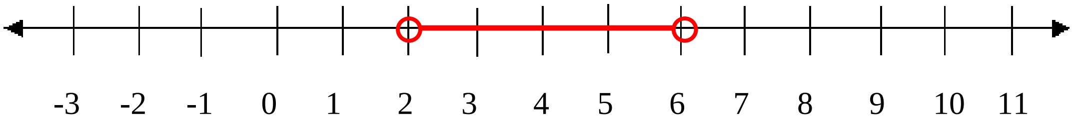
Steps	Reasons
$3\,(2x-9)+2\geq 17-2\,(x+5)$	
$6x-27+2\geq 17-2x-10$	Simplify both sides of the inequality.
$6x-25\geq 7-2x$	
$6x+2x\geq 7+25$	Add 25 and add 2x to both sides of the inequality.
$8x\geq 32$	
$\frac{8x}{8}\geq\frac{32}{8}$	Dividing both sides by a positive number does not switch the inequality.
$x\geq 4$	
$\{x\|\geq 4\}$	We can write the answer using set-builder notation.

We can also graph the inequality:



We use a closed circle to show that the 4 is included.

Sometimes we want to show that a number lies between two values. Consider the following graph:



Because of the open circles 2 and 6 are not included, but all of the values in between are included. There are few ways to express the inequality.

1. $2 < x < 6$
2. $x > 2$ and $x < 6$

We will use the first notation.

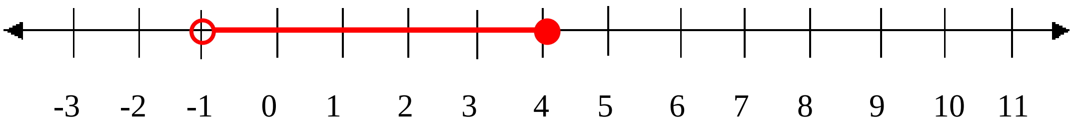
To solve these inequalities we will split the inequality into three parts (which are separated by the inequality) and do the same operations to all three parts at the same time.

Example:

5. Solve $-9 < 4x - 5 \le 11$ write the solution in set notation and graph:

Steps	Reasons
$-9 < 4x - 5 \le 11$ $+5 \qquad +5 \quad +5$	Split the inequality into three parts and try to get x alone in the middle: $-9 \vdots\!< 4x - 5 \vdots\!\le 11$
$-4 < \quad 4x \quad \le 16$	
$\frac{-4}{4} < \frac{4x}{4} \le \frac{16}{4}$	Get x alone in the middle by first adding 5 then dividing by 4 from each of the three parts of the inequality.
$-1 < x \le 4$	
$\{x \mid -1 < x \le 4\}$	Write the answer in set notation.

Graph of the solution:



Examples:

6. In a math class there are four equally weighted tests. A student has grades of 92, 78, and 97 on the first three tests. To earn an A in the class, the students must have an average of at least 90. What must the student get on the fourth test to earn an A in the math class?

Steps	Reasons
Let x = grade on the fourth test.	Use a variable to stand for what we want to find. The question mark indicates what we are looking for.
Average at least 90 Average $\geq$ 90	Start by writing the words then translate the parts. At least means $\geq$
$\frac{92+78+97+x}{4} \geq 90$	To find the average, add the four grades and divide by four.
$\frac{4}{1} \cdot \frac{92+78+97+x}{4} \geq 90 \cdot 4$	To solve first multiply both sides by 4 to get rid of the fractions.
$92+78+97+x \geq 360$	
$267+x \geq 360$	Do the arithmetic and then subtract both sides by 267.
$x \geq 360-267$	The student needs at least a 93 on the fourth test to earn a 90 for math class.
$x \geq 93$	

7. For a product to be labeled as orange juice a state agency requires that at least 65% of the drink is real orange juice. How many ounces of artificial flavors can be added to 26 ounces of real orange juice and have it still be legal to label the drink orange juice?

Steps	Reasons
x = amount of artificial flavors that may be added	Let a variable stand for what is being asked.
Amount of real juice must be at least 65% of total 26 ounces of real juice $\geq$ .65 (x+26) $26 \geq 0.65(x+26)$	At least means more than or equal. The total is the 26 ounces of real orange plus the artificial flavors that are added (x).
$26 \geq 0.65x+16.9$	
$9.1 \geq 0.65x$	Solve the inequality.
$\frac{9.1}{0.65} \geq \frac{0.65x}{0.65}$	
$14 \geq x \text{ or } x \leq 14$	No more than 14 ounces of the artificial flavors can be added.

Exercises

Solve the inequality and graph the solution on a number line:

1. $4x < 8$
2. $3x > 9$
3. $-5x \geq -10$
4. $-2x \leq -4$
5. $x - 7 \leq -6$
6. $x + 4 \geq 7$
7. $x + 2 > -1$
8. $x - 3 < -7$
9. $3 - 4x < 11$
10. $5 - 2x > 9$
11. $3x - 7 < -1$
12. $5x + 7 > -3$
13. $4x - 2 \geq x + 7$
14. $5x + 3 \leq x - 13$
15. $-2x + 2 < x - 7$
16. $2x - 3 > 4x - 7$

Solve and write the answer in set notation:

17. $2(3x - 5) > 5(2x + 4)$
18. $3(2 - 4x) \leq -2(x - 8)$

19. $7 - 3(\mathrm{x} - 3) < 6 - (\mathrm{x} - 4)$

20. $9 - 5(\mathrm{x} + 4) \geq 2 - (\mathrm{x} - 7)$

21. $6\mathrm{x} + 2(4 - 2\mathrm{x}) \geq 8 + 3(\mathrm{x} + 5)$

22. $7 - (\mathrm{x} - 3) + 2\mathrm{x} < 10 - 3(\mathrm{x} - 2)$

23. $3\mathrm{x} - [5\mathrm{x} - (-2)] \geq 2[-5 - (-2\mathrm{x}) + 3^2] - 2$

24. $5 - 3[2\mathrm{x} - (3\mathrm{x} - 2^2)] < 2^2 - 2[3\mathrm{x} - (2\mathrm{x} - 1)]$

25. $-3 < 2x - 1 \leq 5$

26. $-7 \leq 3\mathrm{x} - 4 < 8$

27. $-3 < 5x - 2 < 7$

28. $4 \leq 4\mathrm{x} - 7 \leq 8$

Use an inequality to solve the following:

29. In a math class there are five equally weighted tests. A student has grades of 94, 92, 78, and 97 on the first four tests. To earn an A in the class, the students must have an average of at least 90. What must the student get on the fifth test to earn an A in the math class?

30. For a team competition each member of a four member team runs an obstacle course. In order to pass on to the next level the team must have an average time of at most 60 seconds. If the first three member of the team run the course in 59, 68, and 53 seconds, what times for the fourth member will let the team pass onto the next level?

31. A government agency can spend at most $125,000 on a training program. If the training program has a fixed cost of $45,000 plus a cost of $125 per employee, how many employees can be trained?

32. A large company allotted at most \$75,000 for a training program. If the training program has a fixed cost of \$18,670 plus a cost of \$215 per employee, how many employees can be trained?

33. For a product to be labeled as orange juice a state agency requires that at least 45% of the drink is real orange juice. How many ounces of artificial flavors can be added to 8 ounces of real orange juice and have it still be legal to label the drink orange juice?

34. For a product to be labeled as real juice a state agency requires that at least 15% of the drink is real juice. How many ounces of artificial flavors can be added to 3 ounces of real juice and have it still be legal to label the drink as real juice?

35. The temperatures in Jerez, Spain had the following range in Celsius degrees on a summer day: $20° \le C \le 35°$. The formula to convert Fahrenheit degrees to Celsius is $C = \frac{5}{9}(F - 32)$. Find the temperature range in Fahrenheit degrees for Jerez, Spain on the summer day.

4

We are working with degree two or quadratic expressions ($ax^2 + bx + c$) and equations ($ax^2 + bx + c = 0$). We see techniques such as multiplying and factoring expressions and solving equations using factoring or the quadratic formula.

Course Outcomes:

- Demonstrate mastery of algebraic skills

4.1 Multiplying Algebraic Expressions and FOIL

Multiplication of variables with exponents is reviewed. A monomial is multiplied by a polynomial using the distributive property. Binomials are multiplied by binomials using FOIL (First, Outer, Inner, Last)

4.2 Factoring and Solving by Factoring

Students learn to factor out a common factor and factor trinomials in the forms of $x^2 + bx + c$ or $ax^2 + bx + c$. Solving by factoring and some basic translation problems for degree two equations are covered.

4.3 Quadratic Formula

Students learn to solve degree two equations by factoring and applying the quadratic formula. Quadratic equations may or may not be able to be solved by factoring.

When multiplying monomials and polynomials we add exponents using the following rule:

$$x^m \cdot x^n = x^{m+n}$$

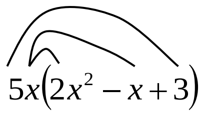
All we are saying is that $x^2 \cdot x^4 = (x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^6$. It is easier to add the 2 and 4 to get the 6 rather than writing it out.

Monomial times a polynomial:

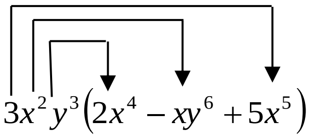
Multiply each term in the polynomial by the monomial by multiplying the numbers and adding the exponents.

Examples

1. Multiply: $5x(2x^2 - x + 3)$

Steps	Reasons
$5x(2x^2 - x + 3)$	Use the distributive property.
	Multiply each term in the polynomial by $5x$: 1. Multiply numbers (coefficients). 2. Multiply each variable by adding exponents, where x has exponent of 1.
$(5x)(2x^2) - (5x)(x) + (5x)(3)$	Here the distributive property is written out. We can really leave this step out by thinking of arrows.
$10x^3 - 5x^2 + 15x$	

2. Multiply: $3x^2y^3(2x^4 - xy^6 + 5x^5)$

Steps	Reasons
$3x^2y^3(2x^4 - xy^6 + 5x^5)$	Use the distributive property.
	Multiply each term in the polynomial by $3x^2y^3$: 1. Multiply numbers (coefficients). 2. Multiply each variable by adding exponents.
$(3x^2y^3)(2x^4) - (3x^2y^3)(xy^6) + (3x^2y^3)(5x^5)$	Here the distributive property is written out. We can really leave this step out by thinking of arrows.
$6x^6y^3 - 3x^3y^9 + 15x^7y^3$	Write the answer.

3. Multiply: $-2y^2\left(3y^5-4y^4-y^3\right)$

Steps	Reasons
$-2y^2\left(3y^5-4y^4-y^3\right)$	Use the distributive property.
$\left(-2y^2\right)\left(3y^5\right)+\left(2y^2\right)\left(4y^4\right)+\left(2y^2\right)\left(y^3\right)$	Multiply $\left(-2y^2\right)$by each term. At least take care of your negative signs and subtraction. A negative number multiplied by a number being subtracted becomes addition. With practice, this step may be skipped by drawing arrows to indicate the correct multiplication.
$-6y^7+8y^6+2y^5$	Multiply the numbers. Multiply the variables by adding their exponents.

The FOIL method for multiplication of binomials is very important in algebra. FOIL is an acronym for:

First: Multiply the first terms of each binomial.
Outer: Multiply the outer terms of the binomials.
Inner: Multiply the inner terms of the binomials.
Last: Multiply the last terms of the binomials.

Examples

4. Multiply: $(x+3)(x+5)$

Steps	Reasons
$(x+3)(x+5)$	To multiply two binomials use FOIL
x^2	First: Multiply the first terms of each binomial.
5x	Outer: Multiply the outer terms of the binomials.
3x	Inner: Multiply the inner terms of the binomials.
15	Last: Multiply the last terms of the binomials.
$x^2+3x+5x+15$	Put all the terms together.
$x^2+8x+15$	Collect like terms.

This problem should be done in three steps since the FOIL part can be done mentally.

5. Multiply: $(3x-7)(2x+5)$

Steps	Reasons
$(3x-7)(2x+5)$	Use FOIL to do all the multiplications mentally. You can draw arrows between the first, outer, inner, last terms to help focus on the multiplication.
$6x^2+15x-14x-35$	First: $3x\cdot 2x=6x^2$ Outer: $3x\cdot 5=15x$ Inner: $-7\cdot 2x=-14x$ Last: $-7\cdot 5=-35$
$6x^2+x-35$	Collect like terms.

Students may wonder where FOIL comes from. Really we are using the distributive property twice. If we do example 4 using the distributive property we will get the same answer.

Steps	Reasons
$(x+3)(x+5)$	Distribute the whole expression $(x+3)$
$(x+3)x+(x+3)5$	Distribute the x and the 5.
$x\cdot x+3x+5x+15$	We end up with the same operations and answers as if we did FOIL
$x^2+8x+15$	

It is easier to multiply these binomials by FOIL. So, we generally do these problems using FOIL.

Exercises

Multiply:

1. $5x(2x^2 + 3x - 4)$
2. $2x(3x^2 - 4x + 5)$
3. $7x(3x^2 - 5x - 11)$
4. $4x(3x^2 - 2x - 9)$
5. $-3x^2(2x^2 - 3x - 6)$
6. $-7x^2(3x^2 - 5x - 9)$
7. $-4x^2(2x^2 - 5x - 6)$
8. $-6x^2(9x^2 - 2x + 5)$
9. $3x^4(5x^4 - 3x^3 - 2x^2)$
10. $7x^5(9x^5 - 3x^4 - 7x^3)$
11. $4xy(3x^2 - 2xy + 5y^2)$
12. $7xy(3y^2 - 5xy - 11y^2)$
13. $-3x^2y(15xy^2 - 10x^2y - 8x^2y^2)$
14. $-7xy^2(12x^2y^2 - 9x^2y + 6xy^2)$
15. $8x^2y(5x^2y - 10xy - 7xy^2 - 12x^2y^2)$
16. $5xy^2(4x^2 - 6xy^2 - 8x^2y - 9x^2y^2)$
17. $(3x + 2)(5x + 4)$
18. $(2x + 1)(3x + 4)$

19. $(5\mathrm{x}-2)(3\mathrm{x}+7)$

20. $(3\mathrm{x}+8)(2\mathrm{x}-9)$

21. $(7\mathrm{x}-2)(5\mathrm{x}-11)$

22. $(4\mathrm{x}+2)(3\mathrm{x}+8)$

23. $(7\mathrm{x}-3)(6\mathrm{x}-5)$

24. $(3\mathrm{x}-11)(2\mathrm{x}-9)$

25. $(11\mathrm{x}-6)(5\mathrm{x}-4)$

26. $(10\mathrm{x}-7)(8\mathrm{x}-6)$

27. $(9\mathrm{x}+6)(8\mathrm{x}-7)$

28. $(8\mathrm{x}-7)(9\mathrm{x}+6)$

29. $(5\mathrm{x}-10)(3\mathrm{x}+12)$

30. $(7\mathrm{x}-15)(4\mathrm{x}-10)$

31. $(3\mathrm{x}^2-9)(2\mathrm{x}^2+4)$

32. $(5\mathrm{x}^2-12)(3\mathrm{x}^2-15)$

33. $(8\mathrm{x}^2-14)(9\mathrm{x}^2-10)$

34. $(7\mathrm{x}^2+8)(12\mathrm{x}^2-9)$

To factor is to write an expression as the product of its factors. The number 6 can be factored into 2⋅3

Factoring out a common factor is based on doing the distributive property in reverse.

$$a\cdot b + a\cdot c = a(b + c)$$

Look for the common factor in all terms:

1. Look for a number that is a common factor.
2. Look for the variables that appear in all terms and use the smallest exponent.

Examples

1. Factor: $15x^5 + 25x^4 - 10x^3$

Steps	Reasons
	5 is a factor of all three terms
$15x^5 + 25x^4 - 10x^3$	x^3 is a factor of all three terms. (Use the smallest exponent.)
$5x^3(3x^2 + 5x - 2)$	Divide each term by the $5x^3$. Actually, I think of what I need to multiply $5x^3$ by to get each of the original terms of the $15x^5 + 25x^4 - 10x^3$

You can check by multiplying:

$$5x^3(3x^2 + 5x - 2) = 15x^5 + 25x^4 - 10x^3$$

2. Factor: $12x^2y^2 - 8x^2y - 40x^2$

Steps	Reasons
	4 is a factor of all three terms
$12x^2y^2 - 8x^2y - 40x^2$	x^2 is a factor of all three terms.
$4x^2(3y^2 - 2y - 10)$	Divide each term by the $4x^2$. I think of what I need to multiply $4x^2$ by to get each of the original terms of the $12x^2y^2 - 8x^2y - 40x^2$

You can check by multiplying.

Factoring polynomials of the form $x^2 + bx + c$

factors of "c" whose sum is "b"

write those numbers (x …)(x…)

After we do an example we can see why it works by checking.

3. Factor x^2+5x+6

Steps	Reasons
x^2+5x+6	b is 5 and c is 6
$(x+2)(x+3)$	Look for the factors of "c" whose sum is "b". List factors of 6 1·6, -1·(-6), 2·3, -2·(-3) 2 and 3 have a sum of 5 Write those numbers after the (x …)(x …)

When we check by multiplying, we see why the sum of the numbers is 5 and the product is 6.

$(x+2)(x+3)$	In the FOIL we will always get x·x
$x\cdot x+3x+2x+6$	2x + 3x = 5x We needed the sum to be 5
x^2+5x+6	2·3 = 6 We only tried factors of 6

4. Factor: $x^2-5x-24$

Steps	Reasons
$x^2-5x-24$	b is –5 and c is –24
List factors of -24 –2·12 , –3·8, 3·(–8)	Look for the factors of "c" whose sum is "b".
3 and –8 have a sum of –24	
$(x-8)(x+3)$	Write those numbers after the (x …)(x …)

5. Factor: x^2+3x+7

Steps	Reasons
x^2+3x+7	b is 3 and c is 7
List factors of 7 1·7, –1·(–7)	Look for the factors of "c" whose sum is "b". None of the factors of 7 add up to 3

So, x^2+3x+7 does not factor. We can also say that x^2+3x+7 is prime.

6. Factor: $6x^3 - 12x^2 - 18x$

Steps	Reasons
$6x^3 - 12x^2 - 18x$	First factor out the common factor.
$6x(x^2 - 2x - 3)$	The factors of –3 whose sum is –2 are – 3 and 1.
$6x(x-3)(x+1)$	Keep the common factor of 6x.

Factoring polynomials of the form $ax^2 + bx + c$ by Guess and Check:

We are doing FOIL backwards. ax^2 is coming from the first terms and c is coming from the last terms. We can check all possibilities that will give us **ax^2** and **c**. Then we check the FOIL to see if we have the right **bx** term.

Examples

7. Factor $2x^2 + 7x + 3$

Steps	Reasons
$(2x \quad)(x \quad)$	$2x^2$ must factor into 2x and x in the first two terms.
Check: $(2x+3)(x+1)$ $(2x+1)(x+3)$ $(2x-1)(x-3)$ $(2x-3)(x-1)$	The factors of 3 are 1,3 and . Try them in both possible orders. They would not be negative because the middle term 7x is positive.
$(2x+1)(x+3)$	If you check the FOIL the answer is the second choice.

Check:
$$(2x+1)(x+3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$$

Here we are guessing 2x and x for the first part of the factors because they will give us $2x^2$ in the FOIL. We guess 1 and 3 for the second part of the factors because they will give us 3 in the FOIL. We have to try the various combinations and do the FOIL to see which combination works.

8. Factor: $6x^2 - 23x - 4$

Steps	Reasons
$(3x\quad)(2x\quad)$ *or* $(6x\quad)(x\quad)$	$6x^2$ must factor into 3x and 2x or 6x and x in the first two terms.
$(3x-4)(2x+1)$ $(3x+4)(2x-1)$ $(3x-1)(2x+4)$ $(3x+1)(2x-4)$ $(3x-2)(2x+2)$ $(3x+2)(2x-2)$ $(6x-4)(x+1)$ $(6x+4)(x-1)$ $(6x-1)(x+4)$ $(6x+1)(x-4)$ $(6x-2)(x+2)$ $(6x+2)(x-2)$	The second part of the two factors has -4,1 or 4,-1, or 2,-2 in either order. This leaves us with 12 possibilities to check. Being organized helps but this method works best when the “a” and “c” are prime numbers like in the first example.
$(6x+1)(x-4)$	If you check the FOIL for all of the combinations, it turns out to be the second to the last.

Types of factoring:

1. Factor out any common factor
   - Always try this first.
   - This is the distributive property backwards.

2. Check the form of the polynomial:

   $x^2 + bx + c$
   - factors of "c" whose sum is "b"
   - write those numbers (x …)(x…)

   $ax^2 + bx + c$
   - We are doing FOIL backwards. ax^2 is coming from the first terms and c is coming from the last terms. We can check all possibilities that will give us **ax^2** and **c**. Then we check the FOIL to see if we have the right **bx** term.
   - write those numbers (_x …)(_x…)

yields ax^2 results in c

Now that we can factor, we can solve some equations that have exponents for the variable. If two numbers are multiplied together to get a product of zero, then one of the must be zero. In other words, If a·b = 0 , then either a is zero or b is zero.

Example

9. Solve $(x-3)(x+2)=0$

Steps	Reasons
$(x-3)(x+2)=0$	If the product equals zero, then one of the factors must equal zero.
$x-3=0 \ or \ x+2=0$	Set each factor equal to zero.
$x=3 \quad or \ x=-2$	Solve the two resulting equations.

A quadratic equation has the form ax^2+bx+c where a,b,c are real numbers and $a\neq 0$. The idea is to solve quadratic equations by factoring.

To solve quadratic equations by factoring:
1. Set the equation equal to zero.
2. Factor.
3. Set each factor equal to zero and solve.

Examples

10. Solve: $x^2+4x=5$

Steps	Reasons
$x^2+4x=5$	Since there is an exponent for the variable:
$x^2+4x-5=0$	1. Set the equation equal to zero.
$(x+5)(x-1)=0$	2. Factor.
$x+5=0 \quad x-1=0$	3. Set each factor equal to zero and solve.
$x=-5 \ or \ x=1$	
$x=-5,1$	There are two solutions to the equation.

11. Solve: $x^3-5x^2+6x=0$

Steps	Reasons
$x^3-5x^2+6x=0$	The equation is already set equal to zero. So, just factor the left.
$x\left(x^2-5x+6\right)=0$	x is a common factor.
$x(x-2)(x-3)=0$	The factors of 6 that add up to –5 are –2 and –3.
$x=0 \ \ or \ \ x-2=0 \ \ or \ \ x-3=0$ $x=2 \qquad x=3$	Set each factor equal to zero.
$x=0,2,3$	These are the solutions.

12. Solve: $4x^2-x=5$

Steps	Reasons
$4x^2-x=5$	Since there is an exponent for the variable: 1. Set the equation equal to zero by subtracting 5 on both sides.
$4x^2-x-5=0$	2. Factor using ax^2+bx+c check the following possibilities.
$(4x-5)(x+1)=0$	$(4x-5)(x+1) \qquad (2x-5)(2x+1)$ $(4x+5)(x-1) \qquad (2x+5)(2x-1)$ $(4x-1)(x+5) \qquad (2x-1)(2x+5)$ $(4x+1)(x-5) \qquad (2x-1)(2x+5)$
$4x-5=0 \ \ or \ \ x+1=0$ $4x=5 \qquad x=-1$ $x=\frac{5}{4}$	3. Set each factor equal to zero and solve.
$x=-1,\frac{5}{4}$	There are two solutions to the equation.

13. Solve: $(2x+5)(x+1)=-1$

Steps	Reasons
$(2x+5)(x+1)=-1$	Since the equation is not yet set equal to zero, we should multiply on the left using FOIL.
$2x^2+2x+5x+5=-1$	Collect like terms and set the equation equal to zero.
$2x^2+7x+6=0$	Factor using the methods for ax² + bx +c $(x+1)(2x+6)$ $(x+2)(2x+3)$ $(x+6)(2x+1)$ $(x+3)(2x+2)$ $(x-1)(2x-6)$ $(x-2)(2x-3)$ $(x-6)(2x-1)$ $(x-3)(2x-2)$
$(x+2)(2x+3)=0$	
$x+2=0$ $2x+3=0$	Check the possibilities to find the factors.
$x=-2$ $2x=-3$	Set each factor equal to zero and solve.
$x=-2 \; or \; x=-\frac{3}{2}$	
$x=-2,-\frac{3}{2}$	There are two solutions to the equation.

Since we can solve equations that have exponents for the variables, we can do word problems that result in this new type of equation. Below are a couple of examples of translation problems that result in degree two equations.

14. The sum of two numbers is six. The sum of the squares of the two numbers is twenty. Find the two numbers.

Steps	Reasons
Sum of two numbers is 6	List information and identify the question.
Sum of squares of two numbers is 20	
Find the numbers???	
x = one number	Here we have two unknowns.
6 – x = the other number	• Let x = one of them is easy. • Because the sum is 6 the other number is 6 – x Sum – first number (Remember: This is the relation for two numbers adding up to a given sum)

Sum of squares of two numbers is 20 One number^2 + other number^2 = 20 $x^2 \quad + \quad (6 – x)^2 \quad = 20$	Use the other information to write an equation.
$x^2+(6-x)^2=20$	Solve the equation.
$x^2+36-12x+x^2=20$	You can multiply (6 – x)(6 – x) on the side.
$2x^2-12x+36=20$	$(6-x)(6-x)$ $36-6x-6x+x^2$ Use FOIL $36-12x+x^2$
$2x^2-12x+16=0$	
$2(x^2-6x+8)=0$	Set the equation equal to zero and factor. Since x^2-6x+8 does not have a number in front of x^2, we find the factors of 8 whose sum is -6. Use -2 and -4 to factor.
$2(x-2)(x-4)=0$	
$x-2=0 \quad x-4=0$	
$x=2, \quad x=4$	
If x = 2, then 6 – x = 6 – 2 = 4	Find the other number using 6 – x.
The numbers are 2 and 4.	State your answer.

15. The sum of two numbers is thirteen. The product of the two numbers is forty. Find the two numbers.

<u>Steps</u>	<u>Reasons</u>
Sum of two numbers is 13 The product of the two numbers is forty.	List information and identify the question.
Find the numbers???	
x = one number 13 – x = the other number	Here we have two unknowns. • Let x = one of them is easy. • Because the sum is 13 the other number is 13 – x Sum – first number (Remember: This is the relation for two numbers adding up to a given sum)

The product of the two numbers is forty. One number × other number = 40 x times $13-x$ =40	Use the other information to write an equation.
$x(13-x)=40$	Solve the equation.
$13x-x^2=40$	Multiply.
$-x^2+13x-40=40-40$	
$-x^2+13x-40=0$	Set equal to zero.
$-(x^2-13x+40)=0$ $-(x-8)(x-5)=0$	I like to factor out the negative sign so that the first term is positive.
$x-8=0$ $x=8$ $x-5=0$ $x=5$	Set each factor equal to zero.
If x = 8 then 13 – x = 13 – 8 = 5	Find the other number using 13 – x.
The numbers are 8 and 5.	State your answer.

Exercises

Factor:

1. $12x^3 + 6x^2 - 18x$
2. $50x^3 + 30x^2 - 70x$
3. $20x^5 - 16x^4 - 12x^3$
4. $21x^5 - 14x^4 - 7x^3$
5. $33x^6 + 55x^4 - 44x^2$
6. $56x^6 - 16x^4 - 32x^2$
7. $63x^5 - 18x^4 + 54x^2$
8. $25y^7 - 15y^6 - 45y^3$
9. $40y^6 - 49y^4 - 35y^2$
10. $11y^7 - 99y^6 - 66y^5$
11. $x^2 + 5x + 6$
12. $x^2 + 4x + 3$
13. $x^2 + 9x + 8$
14. $x^2 + 6x + 8$
15. $x^2 - 2x - 8$
16. $x^2 - 2x + 12$
17. $x^2 + 2x - 7$
18. $x^2 - 3x - 18$

19. $x^2 - 7x + 12$

20. $x^2 - 8x + 15$

21. $x^2 - x - 20$

22. $x^2 - 2x - 3$

23. $x^2 + 4x - 21$

24. $x^2 + 3x - 10$

25. $x^2 - 5x + 4$

26. $x^2 - 12x + 35$

27. $x^2 + 11x + 18$

28. $x^2 + 14x + 40$

29. $x^2 - 5x - 24$

30. $x^2 - x - 42$

31. $x^2 + x - 30$

32. $x^2 + 5x + 2$

33. $x^2 - x - 9$

34. $x^2 + 4x - 12$

35. $x^2 + 16x + 64$

36. $x^2 + 12x + 36$

37. $2x^2 + 5x - 3$

38. $3x^2 + 6x + 2$

39. $3x^2 - 16x + 5$

40. $2x^2 + 5x + 2$

41. $3x^2 - 11x - 10$

42. $3x^2 - 4x + 1$

43. $2x^2 + x - 3$

44. $5x^2 + 9x - 2$

45. $5x^2 - 13x - 6$

46. $2x^2 - 7x + 3$

47. $6x^2 + x - 1$

48. $6x^2 + 5x - 6$

49. $4x^2 - 5x - 6$

50. $4x^2 - 8x + 3$

51. $4x^2 - 4x - 15$

52. $6x^2 - 17x - 3$

Factor Completely:

53. $5x^3 + 20x^2 + 15x$

54. $10x^4 - 10x^3 - 60x^2$

55. $3x^4 - 9x^3 - 30x^2$

56. $7x^5 - 21x^4 + 14x^3$

57. $4x^5 - 24x^4 + 32x^3$

58. $5x^5 + 10x^4 - 40x^3$

59. $20x^5 + 100x^4 + 80x^3$

60. $2x^6 - 16x^5 + 30x^4$

Solve:

61. $(x - 9)(x - 7) = 0$

62. $(x + 6)(x - 5) = 0$

63. $(2x + 3)(3x - 1) = 0$

64. $(5x - 9)(2x + 7) = 0$

65. $x^2 - 5x + 4 = 0$

66. $x^2 + 6x + 5 = 0$

67. $x^2 + 4x - 21 = 0$

68. $x^2 - 3x - 40 = 0$

69. $x^2 = 5x + 6$

70. $x^2 = 6x - 8$

71. $x^2 - x = 12$

72. $x^2 = 2x + 8$

73. $x^2 - 9x = -18$

74. $x^2 - 4x = 21$

75. $x^3 - 3x^2 + 2x = 0$

76. $x^3 + 5x^2 + 6x = 0$

77. $x^3 - x^2 - 20x = 0$

78. $x^3 - 11x^2 + 10x = 0$

79. $3x^2 - 5x - 2 = 0$

80. $2x^2 - 7x + 3 = 0$

81. $2x^2 + x - 3 = 0$

82. $3x^2 + x - 2 = 0$

Solve by using an appropriate equation:

83. The sum of two numbers is eleven. The product of the two numbers is thirty. Find the two numbers.

84. The sum of two numbers is fifteen. The product of the two numbers is thirty-six. Find the two numbers.

85. The sum of two numbers is eight. The sum of the squares of the two numbers is thirty-four. Find the two numbers.

86. The sum of two numbers is seven. The sum of the squares of the two numbers is twenty-nine. Find the two numbers.

Quadratic Formula

Given a polynomial of the form $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This is called the quadratic formula. It is derived by solving for $ax^2+bx+c=0$ by "completing the square," which is beyond the scope of this class.

Examples:

1. Solve $3x^2-2x-5=0$ using the quadratic formula.

Steps	Reasons
$3x^2-2x-5=0$ $a=3,\ b=-2,\ c=-5$	Identify the a,b, and c from the general form: $ax^2+bx+c=0$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Write the quadratic formula.
$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-5)}}{2(3)}$	Replace a,b, and c with the numbers.
$x=\frac{2\pm\sqrt{4+60}}{6}$	Simplify.
$x=\frac{2\pm\sqrt{64}}{6}$	
$x=\frac{2\pm 8}{6}$	
$x=\frac{2+8}{6}=\frac{10}{6}=\frac{5}{3}$ $x=\frac{2-8}{6}=\frac{-6}{6}=-1$	Write out the + and the $-$.
$x=-1,\frac{5}{3}$	Write the answers. The same answer can be found by factoring to solve.

2. Solve $3x^2 = x + 5$ using the quadratic formula.

Steps	Reasons
$3x^2 = x + 5$	Get the polynomial on one side and zero on the other in order to identify the a,b, and c by subtracting x and 5 from both sides. It is usually easier to keep the x^2 term positive.
$3x^2 - x - 5 = 0$	
$a = 3, \quad b = -1, \quad c = -5$	
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the quadratic formula.
$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)}$	Replace a,b, and c with the numbers.
$x = \dfrac{1 \pm \sqrt{1 + 60}}{6}$	Simplify by performing the arithmetic. Careful of the signs!
$x = \dfrac{1 \pm \sqrt{61}}{6}$	The square root cannot be evaluated or simplified. So, we stop.
$x = \dfrac{1 \pm \sqrt{61}}{6} \quad or \quad \left\{ \dfrac{1 - \sqrt{61}}{6}, \dfrac{1 + \sqrt{61}}{6} \right\}$	You may write the answer either way.

The above example shows that we can use the quadratic formula to solve for equations that we cannot solve by factoring. If we are asked to solve a quadratic equation ($ax^2 + bx + c = 0$) and the problem is hard to factor, then we need to try the quadratic formula quickly. It may be that the equation cannot be solved by factoring.

3. Solve $2x^2-4x=7$ using the quadratic formula.

Steps	Reasons
$2x^2-4x=7$	
$2x^2-4x-7=0$	Get the polynomial on one side and zero on the other in order to identify the a,b, and c.
$a=2,\quad b=-4,\quad c=-7$	
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Write the quadratic formula.
$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-7)}}{2(2)}$	Replace a,b, and c with the numbers.
$x=\frac{4\pm\sqrt{16+56}}{4}$	Simplify.
$x=\frac{4\pm\sqrt{72}}{4}$	
$x=\frac{4\pm\sqrt{36}\sqrt{2}}{4}$	Simplify the square root by finding a factor that is a perfect square.
$x=\frac{4\pm6\sqrt{2}}{4}$	
$x=\frac{2\left(2\pm3\sqrt{2}\right)}{4}$	Simplify the fraction by factoring in the numerator and canceling the common factor. We must factor to cancel because of the + or – signs.
$x=\frac{2\pm3\sqrt{2}}{2}$	
$x=\frac{2\pm3\sqrt{2}}{2}\quad or\quad\left\{\frac{2-3\sqrt{2}}{2},\frac{2+3\sqrt{2}}{2}\right\}$	You may write the answer either way.

Exercises

Solve by using the quadratic formula:

1. $3x^2 - 2x - 5 = 0$
2. $2x^2 + 3x - 2 = 0$
3. $5x^2 + 3x - 1 = 0$
4. $4x^2 + 5x = 2$
5. $3x^2 - 7x = -3$
6. $2x^2 - 9x + 5 = 0$
7. $x^2 = 3x - 1$
8. $x^2 = -5x - 1$
9. $3x^2 - x - 7 = 0$
10. $2x^2 + 6x - 9 = 0$
11. $3x^2 - 2x = 9$
12. $4x^2 + 6x - 9 = 0$
13. $12x^2 = 4x + 1$
14. $9x^2 - 3x - 5 = 0$
15. $4x^2 + 2x - 1 = 0$
16. $2x^2 - 3x - 5 = 0$
17. $3x^2 - 4x - 4 = 0$
18. $2x^2 + 10x + 1 = 0$

19. $4\mathrm{x}^2 - 14\mathrm{x} + 3 = 0$

20. $\mathrm{x}^2 - 6\mathrm{x} - 7 = 0$

21. $\mathrm{x}^2 - 2\mathrm{x} - 5 = 0$

5

We formalize the relationship between two variables by defining a function. We explore linear, quadratic, exponential, and logarithmic functions. Students learn to graph these four functions and analyze associated application problems. Systems of linear functions and their applications are studied.

Course Outcome:

- Demonstrate mastery of algebraic skills
- Demonstrate understanding of the concepts of functions and related applications
- Recognize and apply mathematical concepts to real-world situation

5.1 Graphs and Functions

The rectangular coordinate system and ordered pairs are defined. Function notation and the vertical line test for functions are introduced. Linear and parabolic functions are graphed by picking points. The discussion includes applied problems for functions.

5.2 Graphing Linear Equations

Linear equations and three different methods for graphing are explained: picking points, x-intercept/y-intercept, and slope/y-intercept. The slope of a line is described geometrically and algebraically. Vertical lines and horizontal lines are discussed in terms of equation, slope, and graph.

5.3 Solving Systems of Linear Equations

Three methods for solving systems of equations are discussed: graphing, substitution, and addition. Solving by graphing is used to discuss the types of solutions. The focus is on solving using the substitution and addition methods. Applications of solving systems of equations are explored.

5.4 Quadratic Functions

Degree two or quadratic functions are developed. Students will be able to find the vertex, x-intercepts, y-intercept, and graph quadratic functions. Application problems include finding the vertex to determine the minimum or maximum of the quadratic function.

5.5 Exponential Functions

Exponential functions are defined. Students learn to graph exponential functions and solve some applied problems involving exponential functions.

5.6 Logarithmic Functions

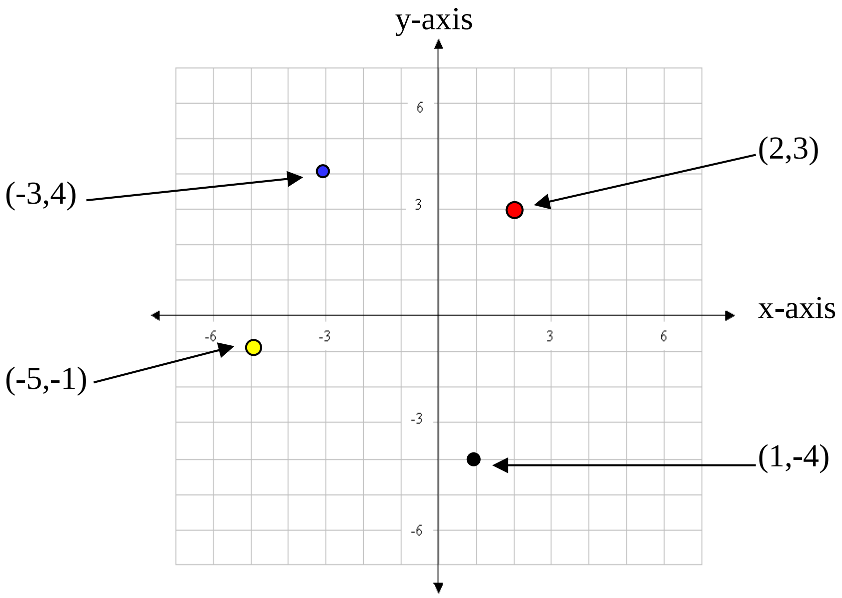
Logarithmic functions are defined. Students should be able to graph logarithmic functions and solve some applied problems involving logarithmic functions.

The idea behind graphing is that we are relating two variables. The rectangular coordinate system is made up of two real number lines that are perpendicular.

Ordered pairs have two coordinates and are used to represent the different points on the coordinate plane:
1. The first coordinate (or abscissa) refers to the horizontal axis (or x-axis).
2. The second coordinate (or ordinate) refers to the vertical axis (or y-axis).

Graphs of Order Pairs

(2,3) 2 to the right on the horizontal axis and 3 up on the vertical axis
(-3,4) 3 to the left on the horizontal axis and 4 up on the vertical axis
(-5,-1) 5 to the left on the horizontal axis and 1 down on the vertical axis
(1,-4) 1 to the right on the horizontal axis and 4 down on the vertical axis



The origin is the point in the middle where the two axes cross with coordinates (0,0).

Function Notation allows us to describe a certain type of relation.
The idea is to define a function f, g, or h in terms of a variable and arithmetic operations. Here we are evaluating functions for a specific value by replacing the variable x with the number and evaluating the expression.

Example

1. Given f(x) = x² +1, find f(1), f(3), and f(-2).

Steps	Reasons
f(1)	
$f(x) = x^2 + 1$	Write the function.
$f(1) = 1^2 + 1$	Replace x with 1.
$f(1) = 1 + 1$	Simplify.
$f(1) = 2$	
f(3)	
$f(x) = x^2 + 1$	Write the function.
$f(3) = 3^2 + 1$	Replace x with 3.
$f(3) = 9 + 1$	Simplify.
$f(3) = 10$	
f(-2)	
$f(x) = x^2 + 1$	Write the function.
$f(-2) = (-2)^2 + 1$	Replace x with -2. Use parentheses to replace the number with the variable.
$f(-2) = 4 + 1$	Simplify.
$f(-2) = 5$	

There really is a great deal that can be said about functions. Functions can be graphed by writing points as $(x, f(x))$ much as we graph with ordered pairs (x, y). To graph we can pick some x-values and then determine their corresponding f(x) values.

Examples

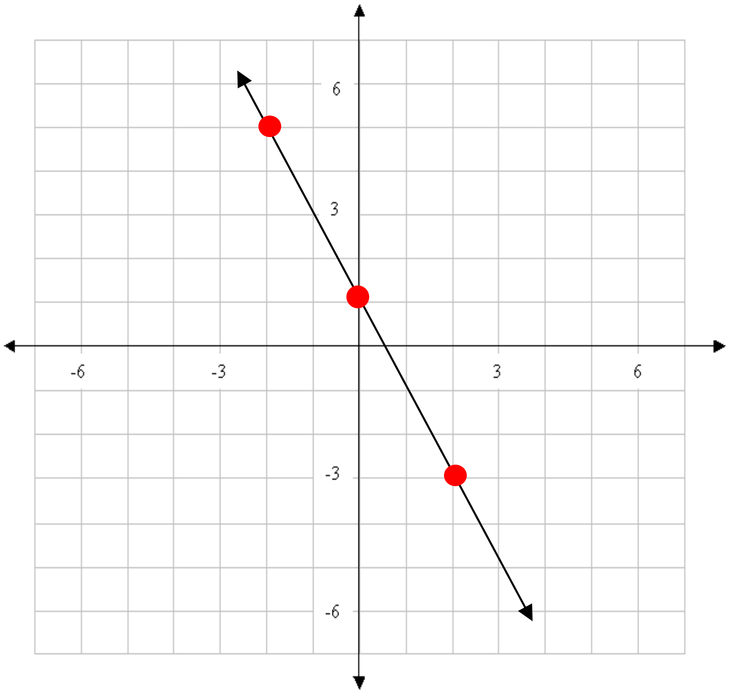
2. Graph $f(x) = -2x + 1$ using x = -2,0,2

x	$f(x)$
-2	5
0	1
2	-3

$$\underline{f(x) = -2x+1}$$
$$f(-2) = -2(-2)+1 = 4+1 = 5$$
$$f(0) = -2(0)+1 = 1$$
$$f(2) = -2(2)+1 = -4+1 = -3$$

Here we have what looks like a linear equation, which we can graph by picking a few points and making a table. Here -2, 0, and 2 were chosen.

Graph the points $(x, f(x))$ from the above table and draw the line that connects them



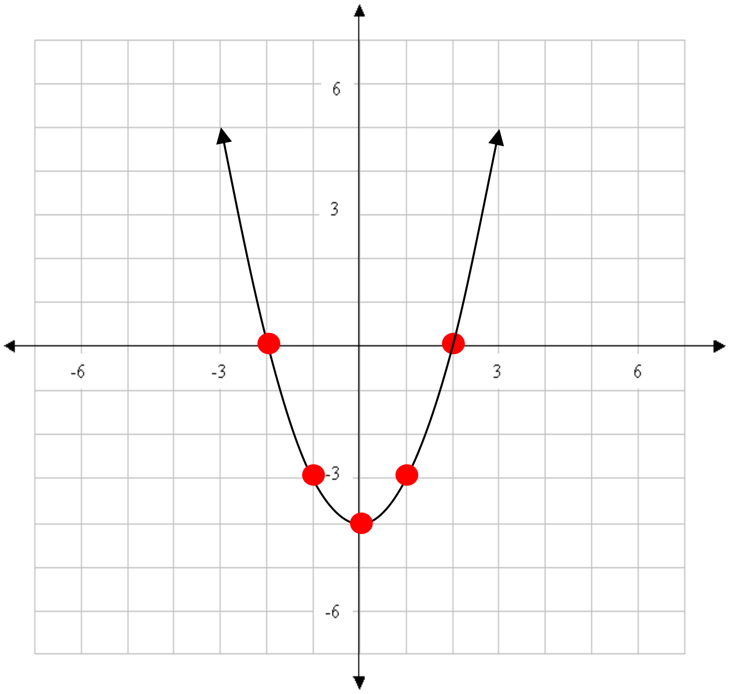
3. Graph $f(x) = x^2 - 4$ using x = -2,-1,0,1,2

x	$f(x)$
-2	0
-1	-3
0	-4
1	-3
2	0

$$f(x) = x^2 - 4$$
$$f(-2) = (-2)^2 - 4 = 4 - 4 = 0$$
$$f(-1) = (-1)^2 - 4 = 1 - 4 = -3$$
$$f(0) = (0)^2 - 4 = 0 - 4 = -4$$
$$f(1) = (1)^2 - 4 = 1 - 4 = -3$$
$$f(2) = (2)^2 - 4 = 4 - 4 = 0$$

We may not know the shape of the graph. Just trying some x-values is a good strategy until we know more about the shape of this type of function. Here we are just trying some x-values around zero: -2, -1, 0, 1, and 2, which turns out to be a very good choice.

Graph the points $\left(x, f(x)\right)$ from the above table.



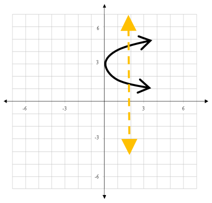
Here the shape is $\cup$, which is called a parabola. It turns out that all <u>quadratic functions</u>, which have the form $f(x)=ax^2+bx+c$, have graphs that are parabolas. The x^2 is the most obvious difference with the linear functions.

Aside from having its own notation, there is a stricter definition of functions that states for every x value there is at most one y value. So, if a graph has two points such as (2,3) and (2,5), it is not the graph of a function because the x=2 is related to two different y-values. From this restriction, we get the following:

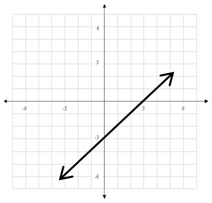
Vertical Line Test:
If no vertical line intersects the graph more than once, then it is the graph of a function.

Examples

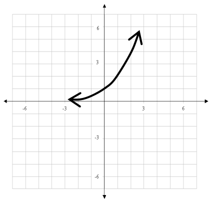
4. Which of the following are graphs of functions?



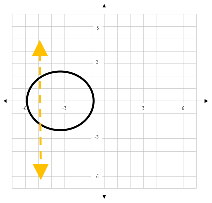
Not a function - we can draw a vertical line that intersects the graph more than once.



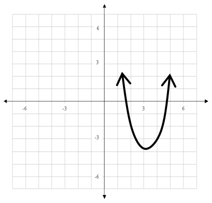
Function - we cannot draw a vertical line that intersects the graph more than once.



Function - we cannot draw a vertical line that intersects the graph more than once.



Not a function - we can draw a vertical line that intersects the graph more than once.



Function - we cannot draw a vertical line that intersects the graph more than once.

5. The cost of a new car can be estimated by the function:
$C(x) = -2.4x^2 + 950x + 8{,}500$ where x is the number of years after 1990.

a. Use the formula to estimate the cost of a new car in 2005.
b. If the actual cost of a new car in 2005 was $21,750, does the formula underestimate or overestimate the actual price? By how much?

Steps	Reasons
$C(x) = -2.4x^2 + 950x + 8{,}500$	2005 – 1990 = 15
$C(15) = -2.4(15)^2 + 950(15) + 8{,}500$	2005 is 15 years after 1990. So, use x=15.
$C(15) = 22{,}210$	Use a calculator to evaluate the function.
a. In 2005 the new car will cost about $22,100.	State the answer.
b. The function overestimates the cost of the new car by $460.	Since $22,100 is more than the actual price of $21,750, the formula overestimates the actual price by 22,100 – 21,750 = 460.

Exercises

Using a straightedge, please draw the x-axis and y-axis on a sheet of graphing paper. Then, plot each of the following.

1. $(2,5)$
2. $(-4,3)$
3. $(-3,-5)$
4. $(5,0)$
5. $(3,-1)$
6. $(-3,2)$
7. $(0,0)$
8. $(-4,-2)$
9. $(2.5,3.5)$
10. $(1.5,-2.5)$
11. $\left(\frac{1}{2},\frac{3}{4}\right)$
12. $\left(-\frac{1}{3},\frac{2}{3}\right)$

Evaluate the function for the following values:

13. $\mathrm{f(x) = 3x - 5}$
    a. $\mathrm{f(2)}$
    b. $\mathrm{f(10)}$
    c. $\mathrm{f(-3)}$
    d. $\mathrm{f(0)}$

14. $\mathrm{h(x) = 8x - 5}$
    a. $\mathrm{h(4)}$
    b. $\mathrm{h(-5)}$
    c. $\mathrm{h}\left(\frac{1}{4}\right)$
    d. $\mathrm{h(0)}$

15. $\mathrm{g(x) = x^2 + 5x}$
    a. $\mathrm{g(2)}$
    b. $\mathrm{g(-3)}$
    c. $\mathrm{g(0)}$
    d. $\mathrm{g}\left(\frac{1}{5}\right)$

16. $f(x) = 2x^2 - 3x$
    a. $f(4)$
    b. $f(-5)$
    c. $f(0)$
    d. $f(-3)$

17. $f(x) = 3x^2 + 8$
    a. $f(3)$
    b. $f(-2)$
    c. $f(0)$
    d. $f(-5)$

18. $g(x) = 5x^2 - 3$
    a. $g(1)$
    b. $g(-2)$
    c. $g(0)$
    d. $g\left(\frac{1}{2}\right)$

19. $h(x) = \sqrt{x + 9}$
    a. $h(7)$
    b. $h(-5)$
    c. $h(0)$
    d. $h(91)$

20. $f(x) = \sqrt{x + 4}$
    a. $f(21)$
    b. $f(-3)$
    c. $f(0)$
    d. $f(285)$

Graph the following functions by first finding the value of the function for the x-values of -2,-1,0,1,2 and then graphing those points. In the end the points should be connected to show the appropriate shape.

21. $f(x) = 3x - 1$

22. $f(x) = 2x + 1$

23. $f(x) = -2x + 1$

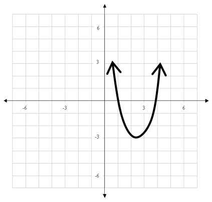
24. $h(x) = \frac{1}{2}x + 3$

25. $g(x) = x^2 + 1$

26. $g(x) = x^2 - 2$

Which of the following are graphs of functions?

27.



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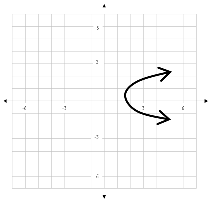

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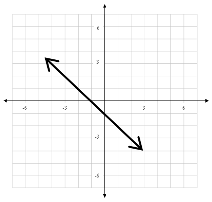

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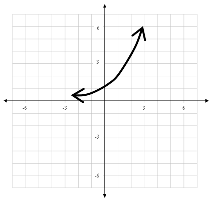

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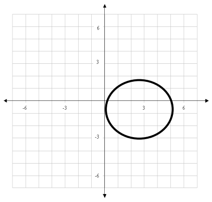

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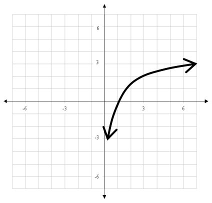

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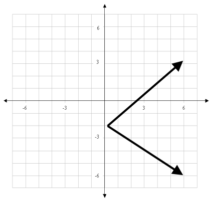

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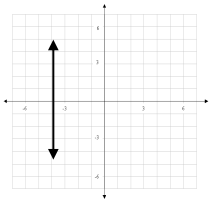

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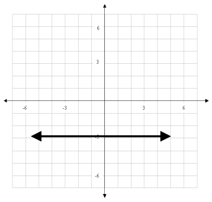


Answer the following:

36. The weight of water in a container is related to the volume of water in the container. A rectangular swimming pool with length 15 meters, width 5 meters, and a constant depth is being filled with fresh water. By constant depth we mean that there is no shallow end or deep end. Instead the height or depth of the water is the same throughout the pool. The weight of the water in kilograms in the pool is $\mathrm{W(x) = 750x}$ where x is the depth of water in the pool in centimeters.
    a. What is the weight of the water when the depth of the water in the pool is 10 cm?
    b. What is the weight of the water in the swimming pool when the depth of the water is 250 cm?

37. The weight of water in a container is related to the volume of water in the container. A rectangular swimming pool with length 15 meters, width 5 meters, and a constant depth is being filled with salt water. By constant depth we mean that there is no shallow end or deep end. Instead the height or depth of the water is the same throughout the pool. The weight of the water in kilograms in the pool is $\mathrm{W(x) = 772.5x}$ where x is the depth of water in the pool in centimeters.
    a. What is the weight of the salt water when the depth of the water is 10 cm?
    b. What is the weight of the salt water in the swimming pool when the depth of the water is 250 cm?

38. The cost of a new computer can be estimated by the function: $C(x) = -1.8x^2 + 80x + 750$ where x is the number of years after 1995.
    a. Use the formula to estimate the cost of a new computer in 2009.
    b. If the actual cost of a new computer in 2009 was $1450, does the function underestimate or overestimate the actual price? By how much?

39. The cost of sending a child to an elite private college can be estimated by the function: $C(x) = 18x^2 + 700x + 2000$ where x is the number of years after 1980.
    a. Use the formula to estimate the cost of sending a child to an elite private college in 2015.
    b. If the actual cost of sending a child to an elite private college in 2015 was $52,000, does the function underestimate or overestimate the actual price? By how much?

Solutions to equations with two variables are ordered pairs. Since there can be an infinite number of solutions it may be impossible to list all the solutions. Instead we may show the solutions by drawing a graph of the solutions on the rectangular coordinate system. To check to see if an individual point is a solution, replace x and y in the equation with the appropriate number from the ordered pair (x,y).

(3,2) is a solution to $5x - 4y = 7$
Check:

$$5(3) - 4(2) = 7$$
$$15 - 8 = 7$$
$$7 = 7 \ \ True$$

(1,-5) is not a solution to $y = 2x - 6$.
Check:

$$-5 = 2(1) - 6$$
$$-5 = 2 - 6$$
$$-5 = -4 \ \ False$$

Examples

1. Is (5,3) a solution to $y = 2x - 7$?

Steps	Reasons
5 is x and 3 is y	Check by evaluating the equation for these values of x and y.
$y = 2x - 7$ $3 = 2(5) - 7$ $3 = 10 - 7$ $3 = 3$	If the replacement is true, then the ordered pair is a solution. If the replacement is false, then the ordered pair is not solution.
Yes, (5,3) is a solution to $y = 2x - 7$	Say yes or no.

When we write equations with two variables, there can be an infinite number of different combinations that are solutions depending on the choice of x. Since all of the solutions cannot be listed, we graph pictures of the solutions called graphs.

Linear equations in two variables are equations whose graphs are lines. Every point on the line will be a solution to the equation.

There are three main ways to graph linear equations. In the beginning we just pick a few points. Two points determine a line and another point will help to check.

Graphing by picking points

1. Pick three x-values. 2,0,-2 or 3,0,-3 are often good choices. Three points around $x=0$ that avoid fractions is best.
2. Calculate the y-value.
3. Graph the points on a set of axes.
4. Draw the line. Use a straight-edge for all lines including axes.

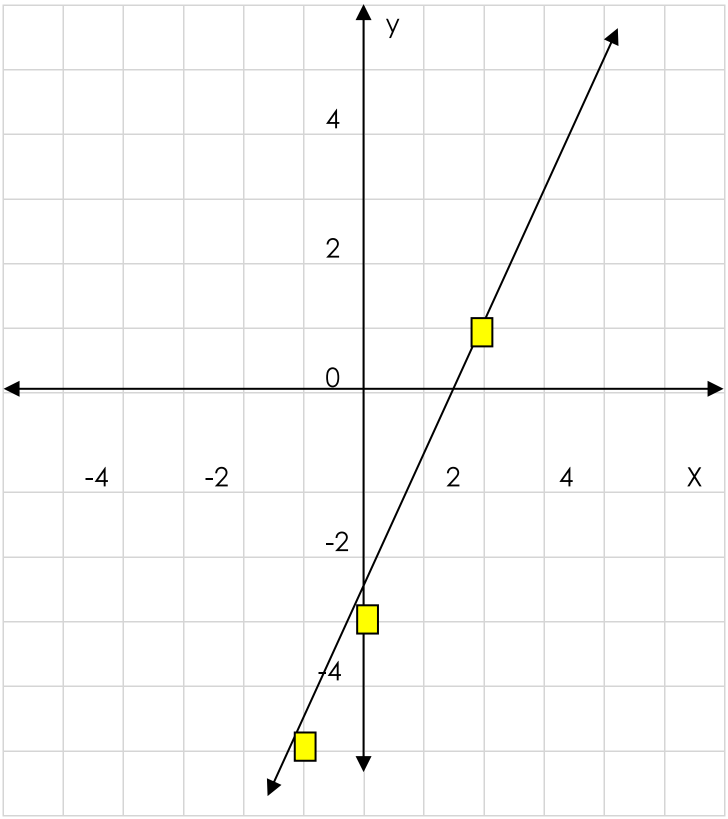
Examples

2. Graph: $y = 2x - 3$

Make a table. You pick the x-values. -2,0,2 were chosen, but other x-values will yield the same line.
Evaluate $y = 2x - 3$ for the x-values that you choose. Note that $y = 2x - 3$. So, evaluating $2x - 3$ for x-values gives y-values and points on the line.

X	Y	$Y = 2x - 3$
-2	-7	$2(-2) - 3 = -4 - 3 = -7$
0	-3	$2(0) - 3 = 0 - 3 = -3$
2	1	$2(2) - 3 = 4 - 3 = 1$
-1	-5	$2(-1) - 3 = -2 - 3 = -5$

From the chart the points $(-2,-7)$, $(0, -3)$, $(2,1)$, and $(-1,-5)$ are all points on the line. There are yellow squares over the points from the graph. Draw the line that connects the points.

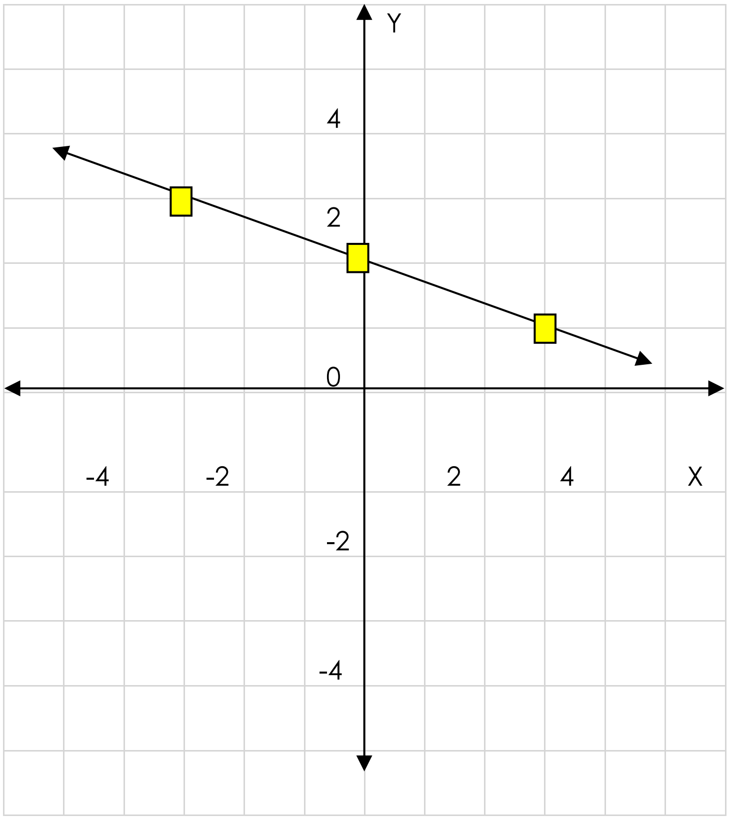


3. Graph $y=-\frac{1}{3}x+2$

X	Y	$y=-\frac{1}{3}x+2$
– 3	3	$-\frac{1}{3}(-3)+2=1+2=3$
0	2	$-\frac{1}{3}(0)+2=0+2=2$
3	1	$-\frac{1}{3}(3)+2=-1+2=1$

Make a table. You pick the x-values. I picked multiples of three (denominator) to avoid fractions. Evaluate $-\frac{1}{3}x+2$ for the x-values that you choose. Note that $y=-\frac{1}{3}x+2$. So evaluating $-\frac{1}{3}x+2$ for x-values gives y-values and points on the line.

The line goes through the points (–3,3), (0,2), and (3,1).

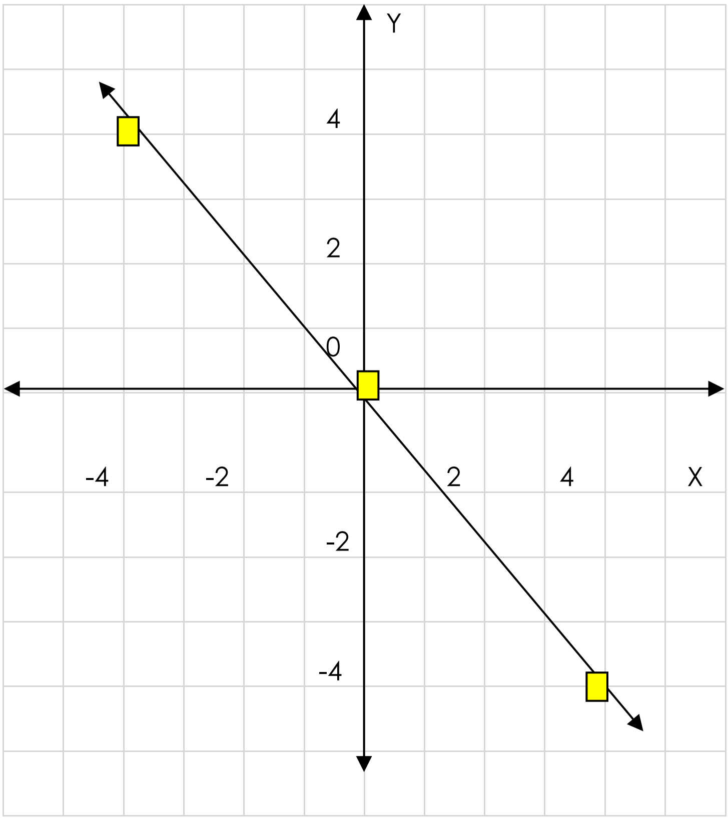


4. Graph: y = –x

X	Y	y = –x
– 4	4	– (– 4) = 4
0	0	–0 = 0
4	– 4	– 4

Make a table. You pick the x-values. Evaluate y = –x for the x-values that you choose.

(–4,4), (0,0), and (4,– 4) are all points on the line.



Because there may be an infinite number of solutions to an equation in two variables, we often draw the solutions. Remember, solutions are ordered pairs. So, the graphs will be on the coordinate plane.

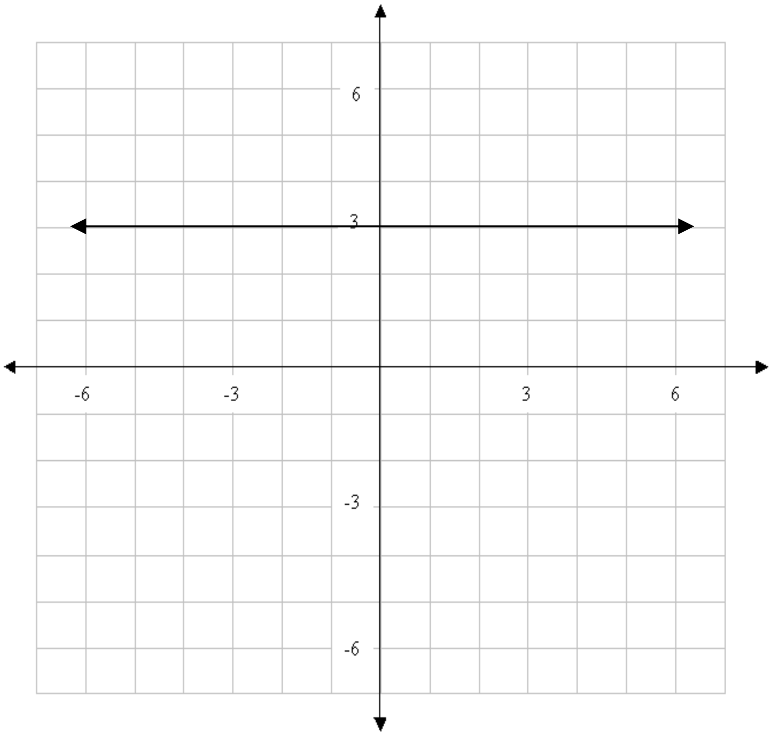
Here we are graphing <u>linear equations</u>, which means our graphs will be lines.

<u>Horizontal lines</u> have the form y = b.

<u>Vertical lines</u> have the form x = a.

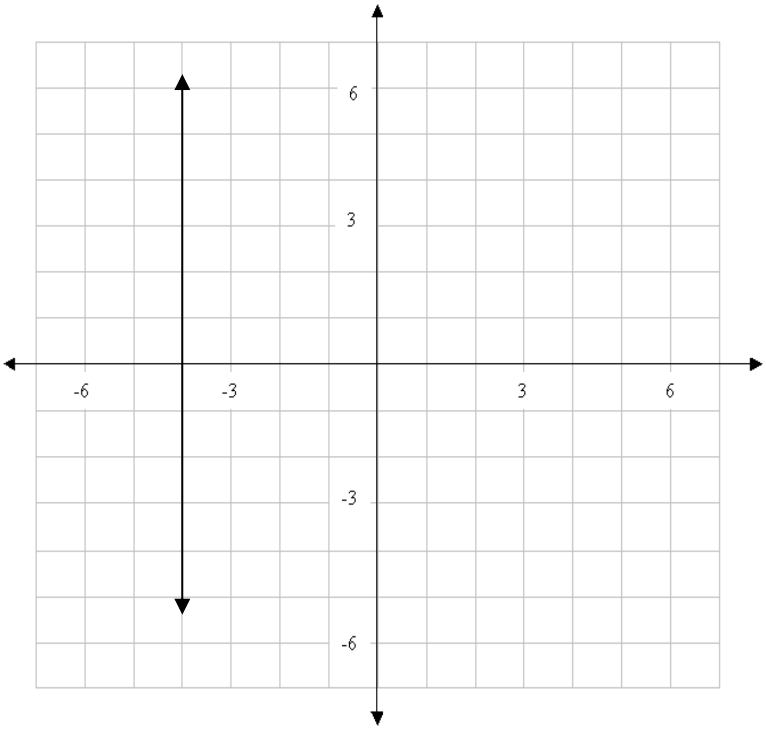
It is best to graph the horizontal and vertical lines by recognizing the form of the equation.

5. Graph y = 3



y = 3 is a horizontal line. Notice that all of the points on the line have a second coordinate of 3. (y=3)

6. Graph x = – 4

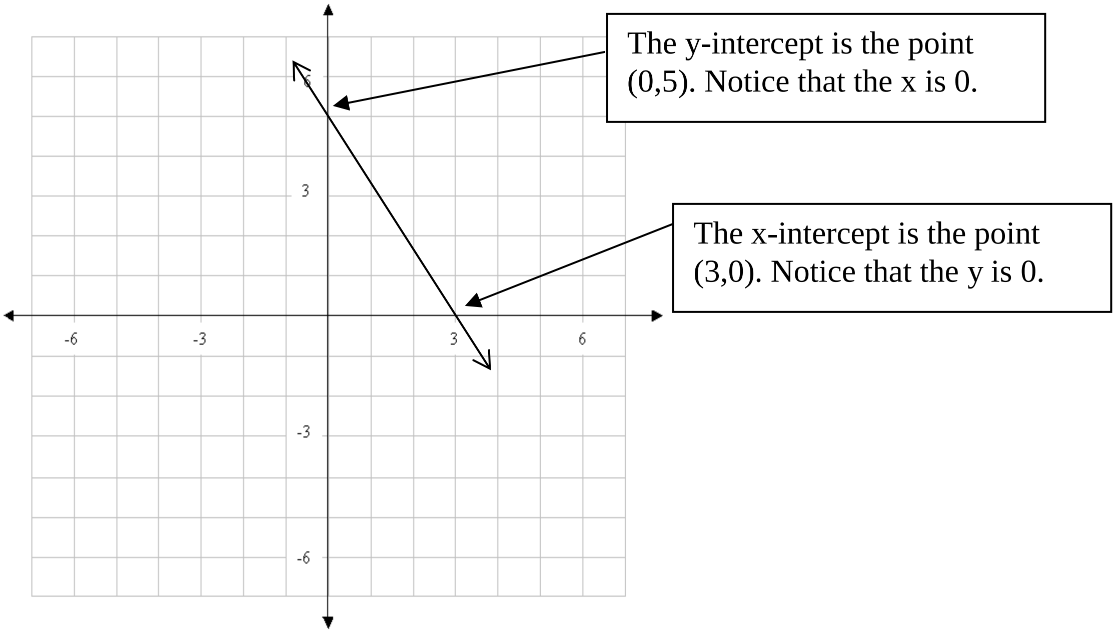


x = – 4 is a vertical line. Notice that all of the points on the line have a first coordinate of – 4. (x = – 4)

The x-intercept is the point where the graph crosses the x-axis. The x-intercept always has second coordinate of 0 (y = 0).

The y-intercept is the point where the graph crosses the y-axis. The y-intercept always has first coordinate of 0 (x = 0).

Look at the x-intercept and y-intercept on the following graph.



The y-intercept is the point (0,5). Notice that the x is 0.

The x-intercept is the point (3,0). Notice that the y is 0.

To find the x-intercept:

1. Let $y = 0$
2. Solve for x.
3. Write answer.

To find the y-intercept:

1. Let $x = 0$
2. Solve for y.
3. Write answer.

Graphing using the x-intercept and y-intercept:

1. Graph the x-intercept and y-intercept.
2. Draw the line through the intercepts.

Vertical and horizontal lines cannot be graphed using this method. Graphing by intercepts works best when the equation has the form $Ax + By = C$.

Examples

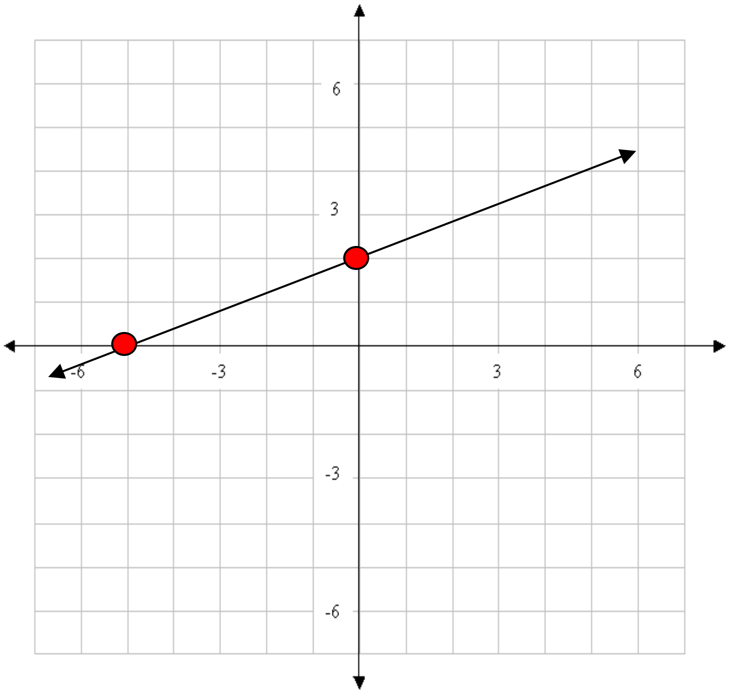
7. Find the x-intercept and y-intercept for $6x - 12y = 24$

x-intercept (Let y=0)	y-intercept (Let x=0)
$6x - 12(0) = 24$	$6(0) - 12y = 24$
$6x = 24$	$-12y = 24$
$x = 4 \ or \ (4,0)$	$y = -2 \ or \ (0,-2)$

8. Find the x- and y-intercepts and graph $2x - 5y = -10$

x-intercept (Let y=0)	y-intercept (Let x=0)
$2x - 5(0) = -10$	$2(0) - 5y = -10$
$2x = -10$	$-5y = -10$
$x = -5 \ or \ (-5,0)$	$y = 2 \ or \ (0,2)$

Graph the intercepts and then graph the line containing two intercepts. Here we have a second way to graph.



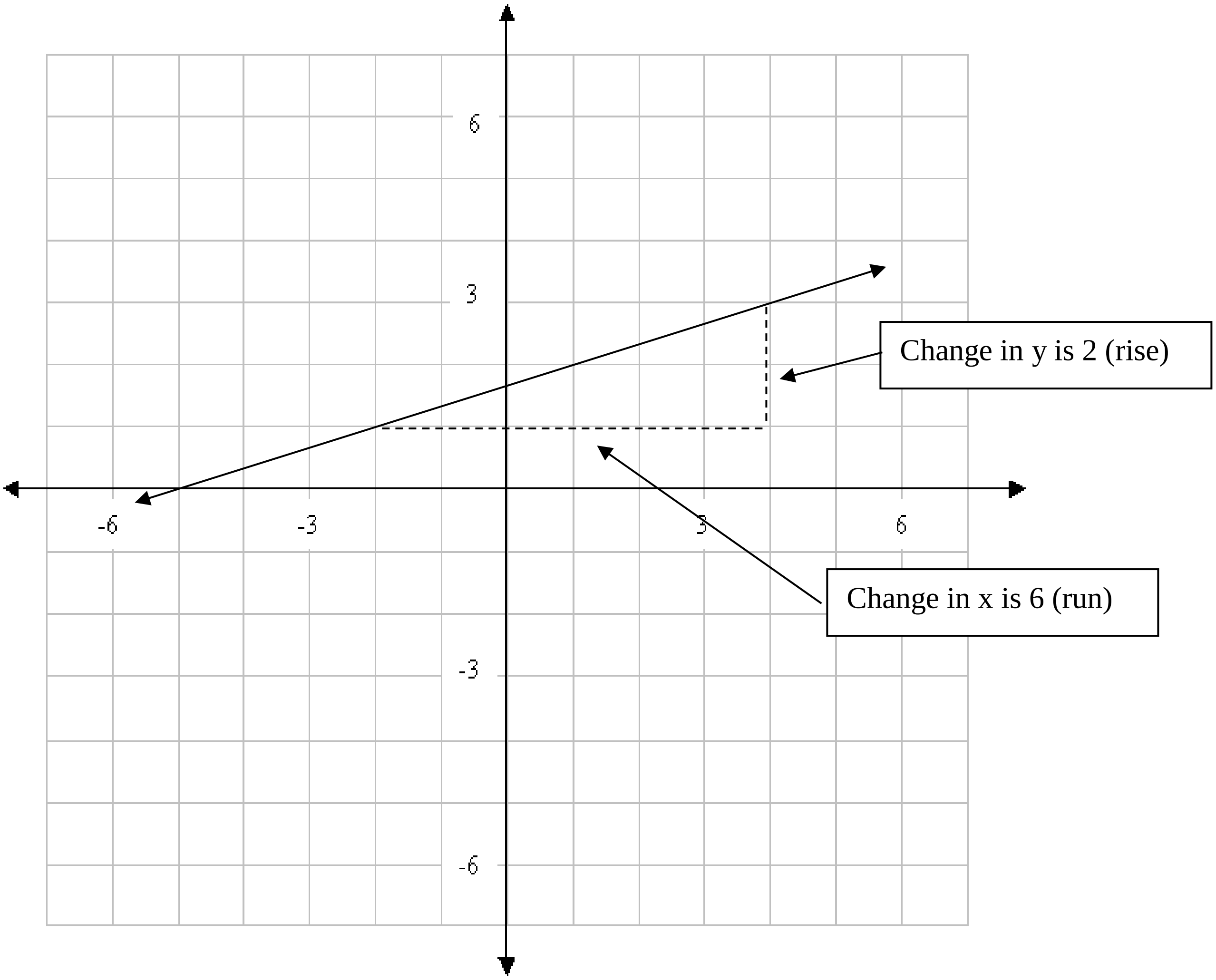
The slope of a line (m) can be thought of as:

1. $m = \frac{the\ change\ in\ y}{the\ change\ in\ x}$

2. $m = \frac{rise}{run}$

3. $m = \frac{y_2 - y_1}{x_2 - x_1}$ for any two points on the line (x_1, y_1) and (x_2, y_2)

<u>Look at any line:</u>



The slope of this line is

$$m = \frac{the\ change\ in\ y}{the\ change\ in\ x} = \frac{2}{6} = \frac{1}{3}$$

<u>Examples:</u>

9. Find the slope of the line containing the points (–2,1) and (4,3).
(x_1, y_1) and (x_2, y_2)

Substitute the x and y values into the formula and simplify.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Note: That the two points were on the line above and we got the same slope. Either by looking at the graph or by looking at two points on the line we get the same slope.

10. Find the slope of the line containing the points (5,1) and (5,– 4).

$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-4-1}{5-(5)}=\frac{-5}{0}$ The slope is undefined.

If you draw the two points, you will see that the points lie on a vertical line.

Vertical lines have undefined slope.

Horizontal lines have slope of zero.

Two lines are parallel lines if they have the same slope (and different y-intercepts.)

A line is in slope-intercept form if it is written as

$y=mx+b$

m = slope
b = y-intercept

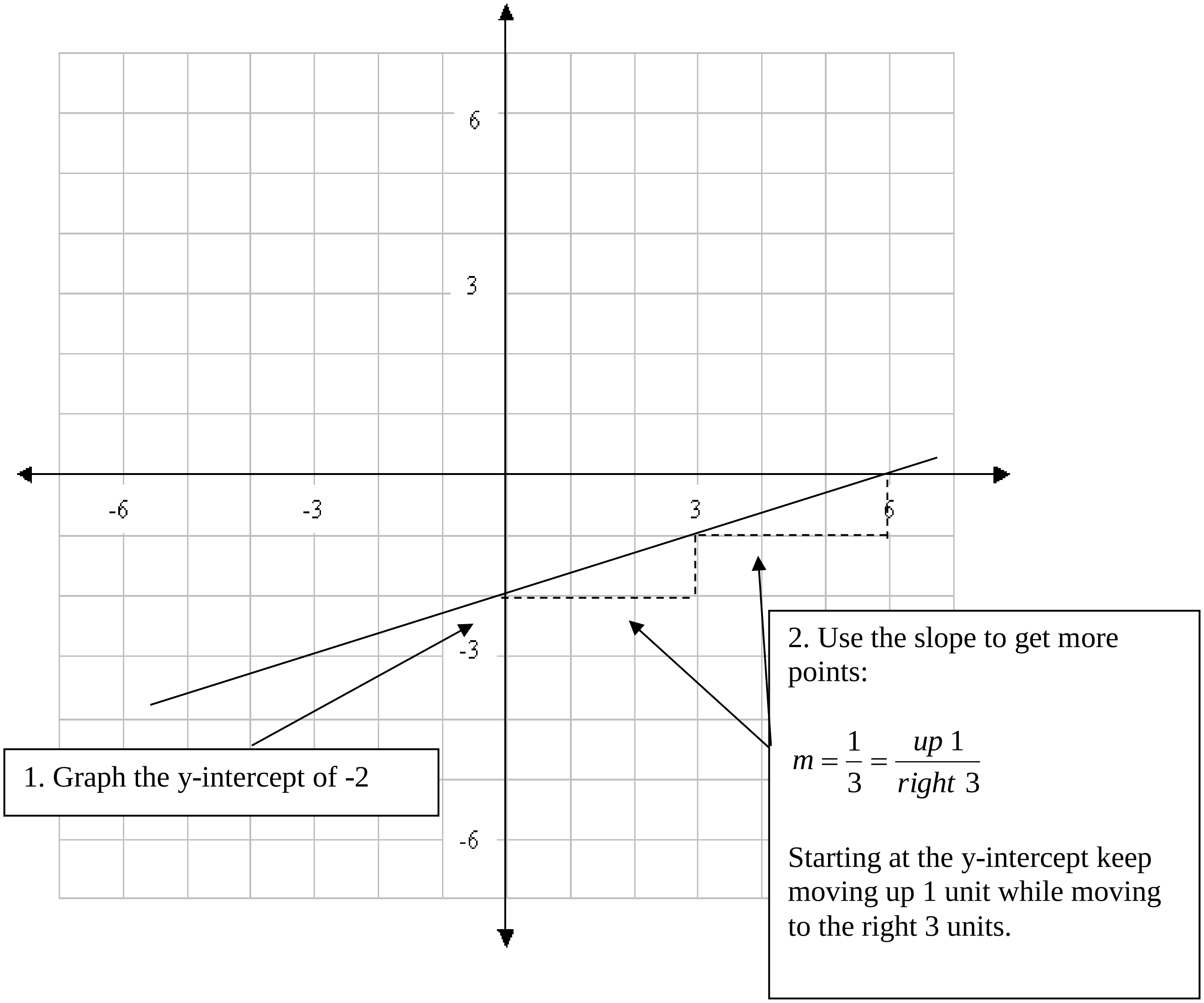
We can read off the information of the slope and y-intercept is in this form. Then we can use that information to graph. Here we have a third way to graph lines.

Steps for graphing using the slope and y-intercept.

1. Get the equation in $y=mx+b$ form to get the slope and y-intercept.
2. Graph the y-intercept.
3. Write the slope as a fraction and use $m=\frac{\text{the change in } y}{\text{the change in } x}$ which is the same as $m=\frac{rise}{run}$ to get more points.

Examples:

11. Graph $y = \frac{1}{3}x - 2$ using the slope and the y-intercept.



1. Graph the y-intercept of -2

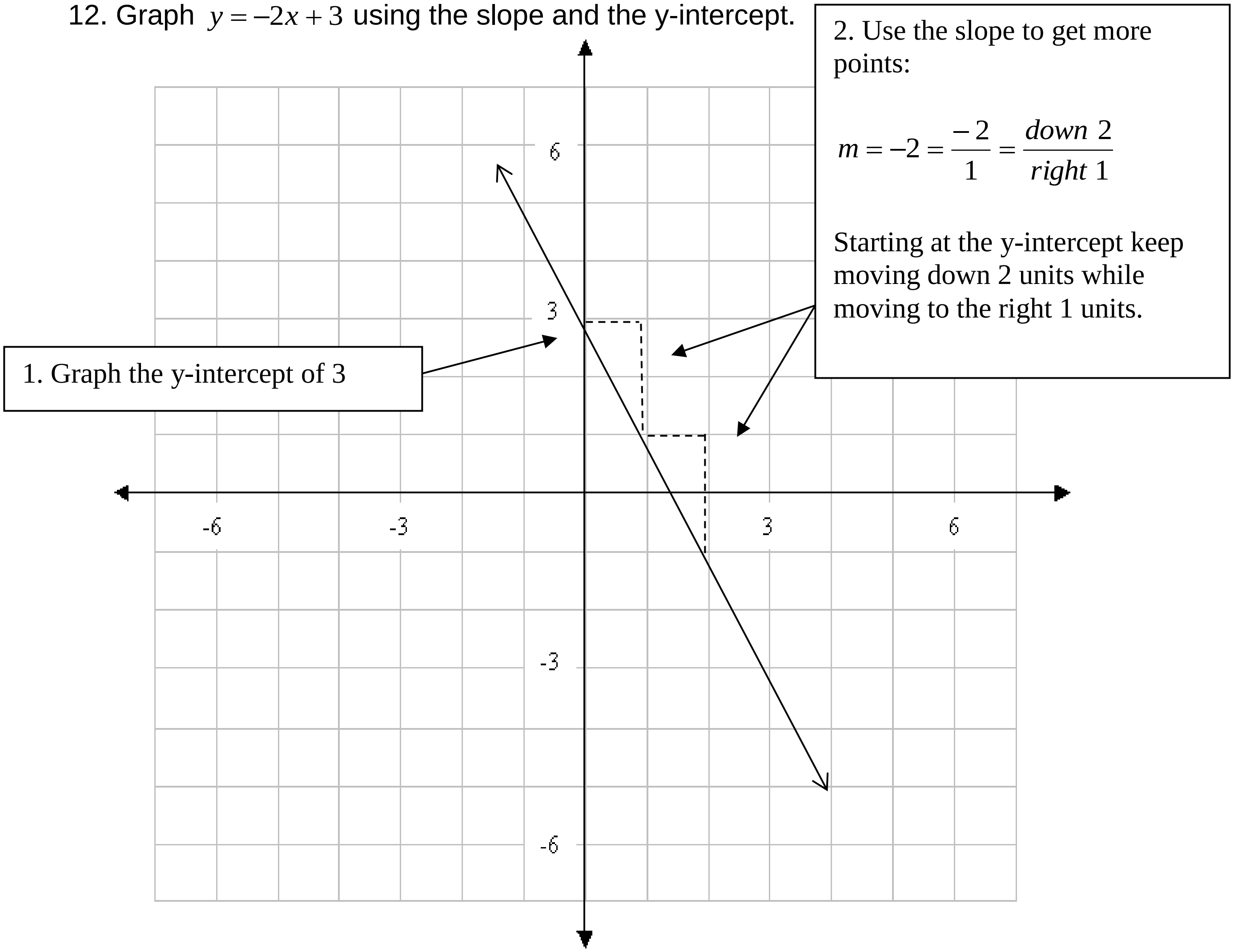
2. Use the slope to get more points:

$$m = \frac{1}{3} = \frac{up\ 1}{right\ 3}$$

Starting at the y-intercept keep moving up 1 unit while moving to the right 3 units.

Because $m = \frac{the\ change\ in\ y}{the\ change\ in\ x}$, we can read off the information for up and down from the numerator and left or right from the denominator.

12. Graph $y = -2x + 3$ using the slope and the y-intercept.



1. Graph the y-intercept of 3

2. Use the slope to get more points:

$$m = -2 = \frac{-2}{1} = \frac{down\ 2}{right\ 1}$$

Starting at the y-intercept keep moving down 2 units while moving to the right 1 units.

There are three ways to graph lines:

1. Graphing by picking points
    1. Pick three x-values. 2,0,-2 or 3,0,-3 are often good choices. Three points around x=0 that avoid fractions is best.
    2. Calculate the y-value.
    3. Graph the points on a set of axes.
    4. Draw the line. Use a straight-edge for all lines including axes.
2. Graphing using the x-intercept and y-intercept:
    1. Graph the x-intercept and y-intercept.
    2. Draw the line through the intercepts.
3. Graphing using the slope and y-intercept.
    1. Get the equation in $y = mx + b$ form to get the slope and y-intercept.
    2. Graph the y-intercept.
    3. Write the slope as a fraction and use $m = \frac{rise}{run}$ or $m = \frac{the\ change\ in\ y}{the\ change\ in\ x}$ get more points.

Exercises

1. Is $(3,5)$ a solution to $y = 2x - 1$?
2. Is $(4,1)$ a solution to $y = 3x - 11$?
3. Is $(5,-4)$ a solution to $y = -2x + 3$?
4. Is $(-3,-2)$ a solution to $y = -3x + 12$?
5. Is $(-2,3)$ a solution to $y = 2x + 7$?
6. Is $(-3,10)$ a solution to $y = -3x + 1$?
7. Is $(-2,5)$ a solution to $3x + 2y = 9$?
8. Is $(7,-6)$ a solution to $2x - 5y = 20$?

Graph by picking points:

9. $y = -2x + 3$
10. $y = \frac{1}{3}x - 2$
11. $y = -\frac{1}{3}x + 2$
12. $y = 3x - 2$
13. $y = \frac{1}{4}x + 2$
14. $y = -3x + 4$

Graph:

15. $x = 3$
16. $y = 5$
17. $y = -2$

18. $x = -3$

19. $y = -4$

20. $x = 2$

Find the x-intercept and y-intercept. Use the intercepts to graph each of the following:

21. $3x - 2y = 6$

22. $5x + 2y = -10$

23. $2x - 4y = -8$

24. $-5x + 3y = -15$

25. $-2x + 3y = 12$

26. $4x - 3y = -12$

27. $y = -3x + 6$

28. $y = \frac{1}{2}x - 2$

29. $y = -\frac{1}{3}x + 6$

30. $y = 2x - 4$

Find the slope of the line containing the two points:

31. $(3,2)$ and $(1,-6)$

32. $(2,1)$ and $(-1,-5)$

33. $(-3,5)$ and $(0,-4)$

34. $(-4,5)$ and $(2,3)$

35. $(4,3)$ and $(-2,6)$

36. $(-5,-3)$ and $(-2,9)$

37. $(3,4)$ and $(-2,4)$

38. $(2,-5)$ and $(3,-5)$

39. $(4,7)$ and $(4,2)$

40. $(3,5)$ and $(3,2)$

Put the equation in $y = mx + b$ form if it is not already and then graph by using the slope and the y-intercept.

41. $\mathrm{y} = \frac{1}{2}\mathrm{x} - 3$

42. $\mathrm{y} = \frac{1}{3}\mathrm{x} + 2$

43. $\mathrm{y} = -2\mathrm{x} + 3$

44. $\mathrm{y} = -3\mathrm{x} + 5$

45. $\mathrm{y} = -\frac{1}{3}\mathrm{x} + 2$

46. $\mathrm{y} = -\frac{1}{2}\mathrm{x} + 3$

47. $\mathrm{y} = 3\mathrm{x} - 2$

48. $\mathrm{y} = 5\mathrm{x} - 4$

49. $3\mathrm{x} - 6\mathrm{y} = 12$

50. $3\mathrm{x} - 2y = 4$

51. $2\mathrm{x} + 3\mathrm{y} = 3$

52. $2\mathrm{x} - 5\mathrm{y} = 10$

A system of linear equations is made up of two equations whose graphs are lines. For instance, they may have the form $\begin{array}{l} 2x + 3y = 5 \\ 5x - 4y = -3 \end{array}$. A point is a solution to the system of equations if it is a solution to both equations.

Examples

1. Is (1,2) a solution to $\begin{array}{l} 2x + 3y = 8 \\ 5x - 4y = -3 \end{array}$?

Steps		Reasons
$2x+3y=8$ $2(1)+3(2)=8$ $2+6=8$ $8=8$	$5x-4y=-3$ $5(1)-4(2)=-3$ $5-8=-3$ $-3=-3$	Check both equations by plugging in the point.
Yes, (1,2) is a solution to the system of equations.		Since (1,2) is a solution to both equations, it is a solution to the system of equations.

Later when we learn how to solve the systems of equations, we can check our answers by plugging in our answer as above.

We will learn to solve systems of equations three ways:
1. Solve by graphing
2. Solve by substitution method
3. Solve by addition method

Remember, graphs are pictures of the solutions. Each point on a graph is a solution to the equation.

To solve by graphing:
1. Graph both equations on the same set of axes.
2. Where the two graphs cross (intersect) there is a solution to the system of equations.

Examples:

2. Solve by graphing: $3x - y = 3$
$2x + y = 2$

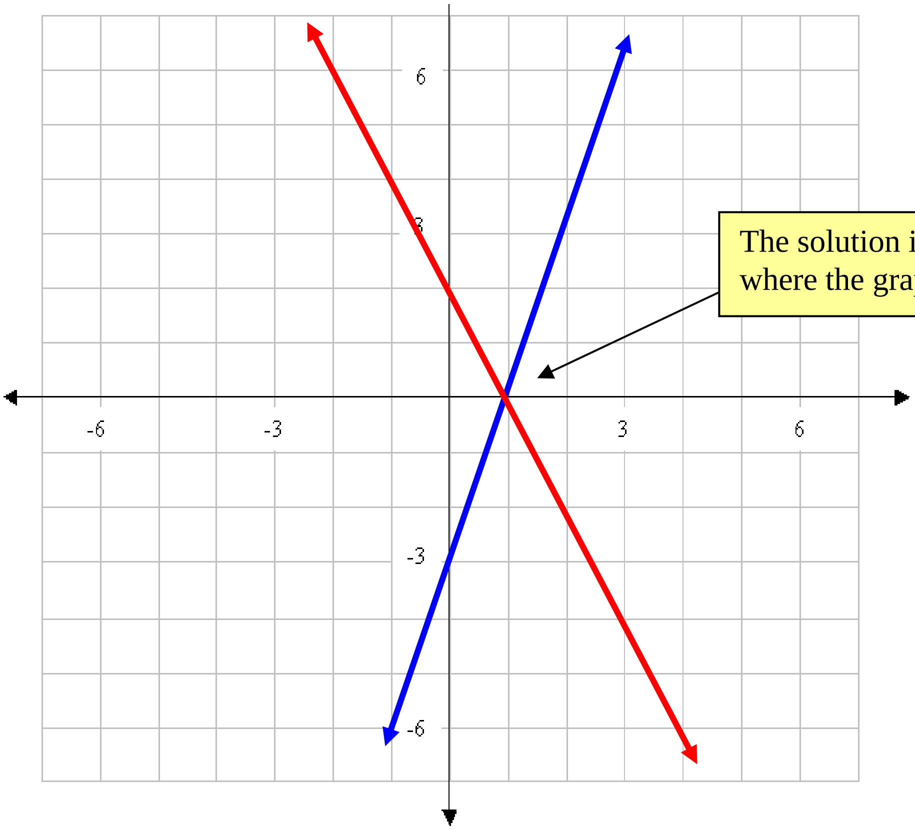
There are several ways to graph each line. Here we are graphing by finding the x-intercept and the y-intercept.

$3x - y = 3$

x-intercept(let y=0)	y-intercept(let x=0)
$3x - 0 = 3$	$3(0) - y = 3$
$3x = 3$	$-y = 3$
$x = 1 \;\; or \;\; (1,0)$	$y = -3 \;\; or \;\; (0,-3)$

$2x + y = 2$

x-intercept(let y=0)	y-intercept(let x=0)
$2x + y = 2$	$2(0) + y = 2$
$2x = 2$	$y = 2$
$x = 1 \;\; or \;\; (1,0)$	$y = 2 \;\; or \;\; (0,2)$



The solution is the point (1,0) where the graphs intersect.

3. Solve by graphing: $\begin{array}{l} x - 4 = 0 \\ y + 2 = 0 \end{array}$

Here we need to recognize that these equations represent a vertical and horizontal line.

$x - 4 = 0$

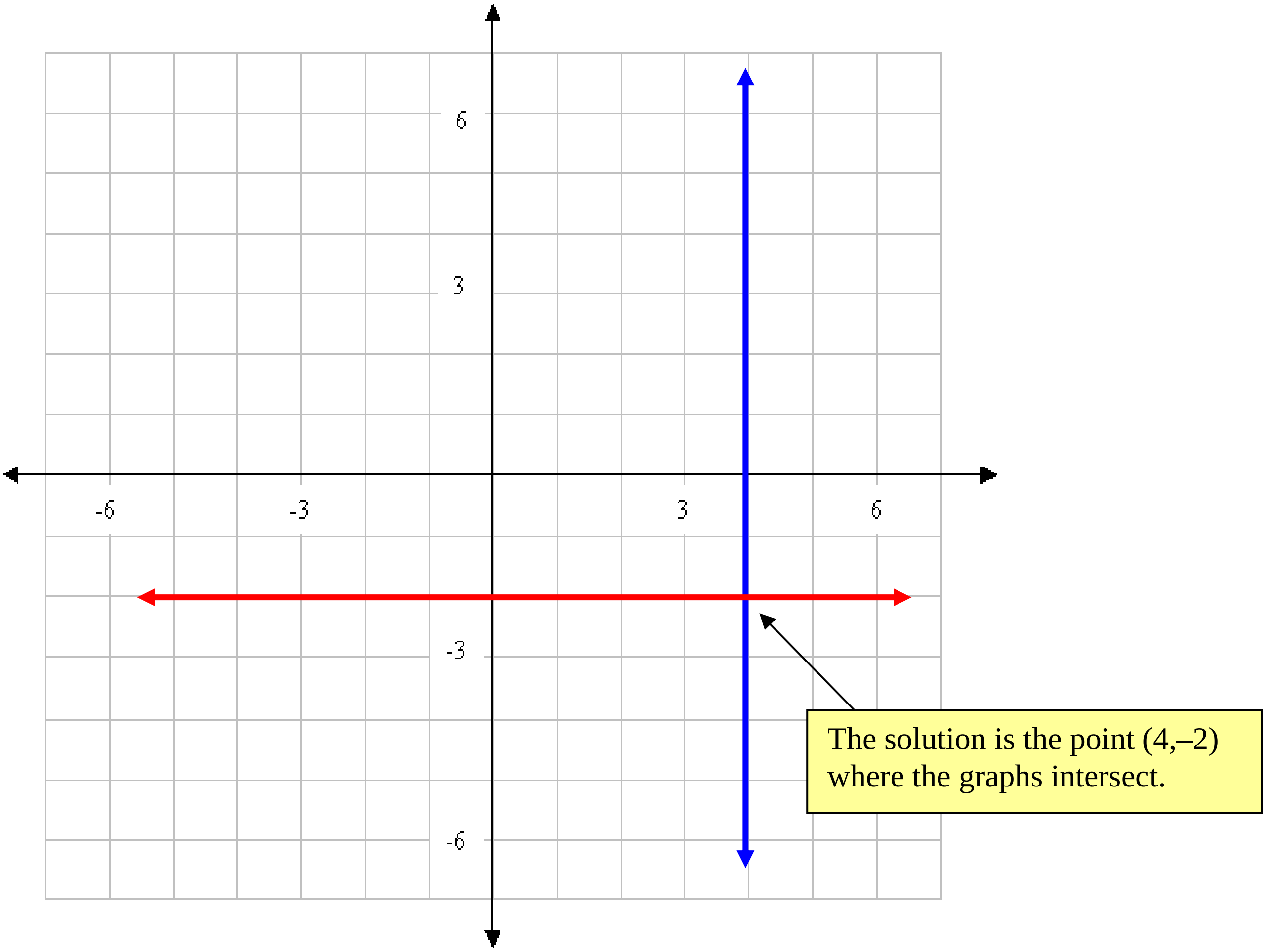
$x = 4$

Equations in this form are vertical lines (all first coordinates are 4).

$y + 2 = 0$

$y = -2$

Equations in this form are horizontal lines (all second coordinates are –2).



The solution is the point (4,–2) where the graphs intersect.

4. Solve by graphing: $2x - y = 6$
$4x - 2y = 4$

There are several ways to graph each line. Below the graphs are found by finding the x-intercept and the y-intercept.

2x – y = 6

x-intercept(let y=0)

$$2x - 0 = 6$$
$$2x = 6$$
$$x = 3 \ \ or \ \ (3,0)$$

y-intercept(let x=0)

$$2(0) - y = 6$$
$$-y = 6$$
$$y = -6 \ \ or \ \ (0,-6)$$

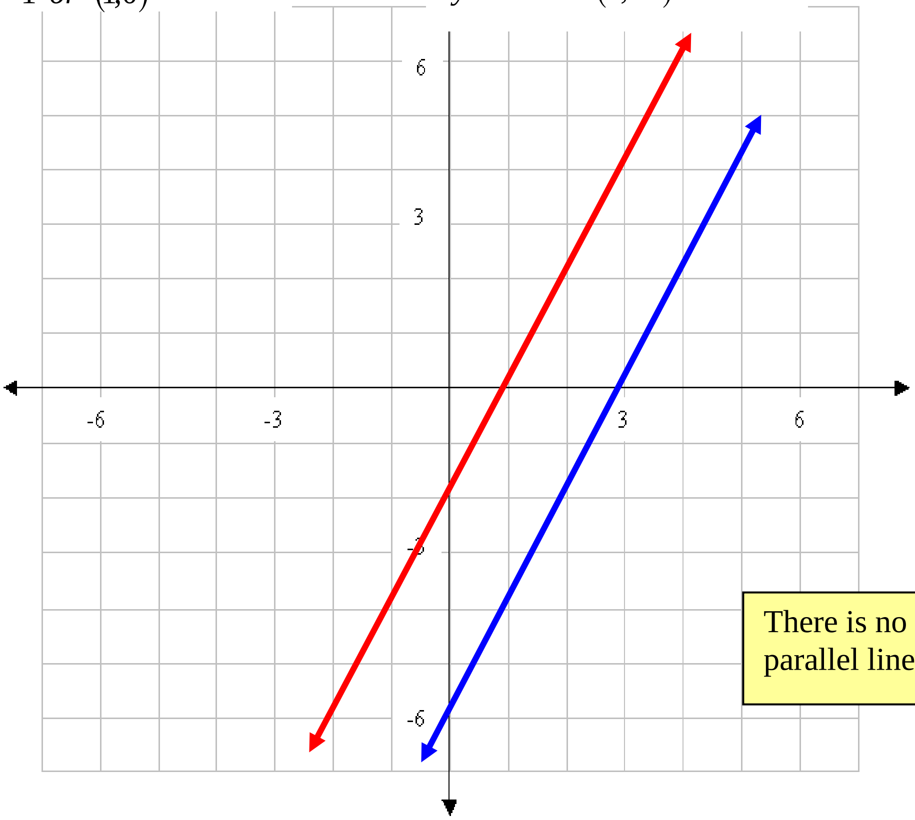
4x – 2y = 4

x-intercept(let y=0)

$$4x - 2(0) = 4$$
$$4x = 4$$
$$x = 1 \ \ or \ \ (1,0)$$

y-intercept(let x=0)

$$4(0) - 2y = 4$$
$$-2y = 4$$
$$y = -2 \ \ or \ \ (0,-2)$$



There is no solution because these parallel lines never intersect.

There are three types of systems of equations. We can relate them to the three possibilities for graphing linear equations.

3 Types of Solutions:

1. **Independent System of Equations:**

1 solution → The graphs intersect at one point.

2. **Dependent System of Equations:**

Line of solutions → The two equations turn out to be the same equation with the same graph.

3. **Inconsistent System of Equations:**

No solutions → The lines are parallel and never intersect.

Solving equations by graphing can be awkward, time consuming, and inexact. There are two other methods that we can use for solving systems of equations:
1. Substitution Method
2. Addition Method

Steps for solving by the substitution method:

1. Solve either equation for either variable. (Pick a variable without a number in front or you may get fractions.)
2. Substitute for the variable in the other equation.
3. Solve the resulting equation.
4. Substitute to get the other variable.
5. Check by plugging in your answer to both equations.

Examples:

5. Solve by the substitution method: $\begin{aligned} 2x + 3y &= -4 \\ x + 4y &= 3 \end{aligned}$

Steps	Reasons
$2x+3y=-4$ $x+4y=3$	Solve the second equation for x. Since there is no number in front of the variable, it will be the best choice.
$x+4y=3$ $x=3-4y$	Subtract 4y from both sides of the second equation. By solving the second equation for x, we never need to divide, which usually introduces fractions.

$2x + 3y = -4$

Replace x in the first equation with 3 – 4y. Now there is only one variable.

$2(3 - 4y) + 3y = -4$

Solve the resulting equation.

$6 - 8y + 3y = -4$

$6 - 5y = -4$

$-5y = -4 - 6$

$-5y = -10$

$\frac{-5y}{-5} = \frac{-10}{-5} \; or \; 2$

$x = 3 - 4y$

Get the other variable by replacing x in either equation.

$x = 3 - 4(2)$

$x = 3 - 8 = -5$

$(-5,2)$ or x = -5, y = 2

State the answer either way.

Check by replacing the variables in both equations.

$2x + 3y = -4$	$x + 4y = 3$	Both equations check.
$2(-5) + 3(2) = -4$	$-5 + 4(2) = -4$	
$-10 + 6 = -4$	$-5 + 8 = -3$	$(-5,2)$ is the solution.
$-4 = -4$	$-3 = -3$	

6. Solve by the substitution method:

$$\begin{aligned} 3x - y &= 7 \\ 2x + 3y &= 1 \end{aligned}$$

Steps	Reasons
$3x - y = 7$ $2x + 3y = 1$	Solve the first equation for y.
$3x - y = 7$	By solving the first equation for y, we never need to divide, which usually introduces fractions.
$3x - 7 = y$	
$y = 3x - 7$	
$2x + 3y = 1$	Replace y in the second equation with 3x – 7. Now there is only one variable.
$2x + 3(3x - 7) = 1$	Solve.
$2x + 9x - 21 = 1$	
$11x = 22$	
$x = \frac{22}{11}$ *or* 2	
$y = 3x - 7$	Get the other variable by replacing x in either equation.
$y = 3(2) - 7$	
$y = 6 - 7 = -1$	
$(2,-1)$ or x = 2, y = –1	State your answer either way.

Check by replacing the variables in both equations.

$3x - y = 7$	$2x + 3y = 1$	Both equations check.
$3(2) - (-1) = 7$	$2(2) + 3(-1) = 1$	$(2,-1)$ is the solution.
$6 + 1 = 7$	$4 - 3 = 1$	
$7 = 0$	$1 = 1$	

7. Solve by the substitution method: $\begin{array}{l} 2x + 4y = 1 \\ x + 2y = 4 \end{array}$

Steps	Reasons
$2x + 4y = 1$ $x + 2y = 4$	Solve the second equation for x.
$x+2y=4$	By solving second equation for x, we never need to divide, which usually introduces fractions.
$x=-2y+4$	
$2x+4y=1$	
$2(-2y+4)+4y=1$	Replace x in the first equation with –2y + 4. Now there is only one variable.
$-4y+8+4y=1$	Solve.
$8=1$ No solution.	8 does not equal 1 regardless of what x and y are.

<u>3 types of answers:</u>
1. The solution is a point: **Independent systems of equations** have a one-point solution. If we graph both lines, they intersect at one point.

2. No solutions. We get a false equality. Above 8 = 1 meant there are no solutions. If we graph both lines, they are parallel and never touch. **(Inconsistent system of equations)**

3. An infinite number of solutions or a line of solutions. If we graph both lines, they are the same line. This occurs if we are left with two numbers that really are equal. If we end up with 5=5 for instance, then we can say there are an infinite number of solutions or a line of solutions. **(Dependent system of equations)**

Another method for solving systems of equations is the addition method:

Steps for solving systems of equations using the <u>addition method</u>:
1. Multiply one or both equations so that on of the variables will cancel after adding the two equations. (For instance 3x + (-3x) would get rid of x's.)
2. Add the equations.
3. Solve for the variable.
4. Replace the number for the variable in either equation to get the other variable.
5. State the answer.
6. Check

Examples:

8. Solve by the addition method: $\begin{array}{l} 4x - 2y = 20 \\ 5x + 2y = 7 \end{array}$

Steps	Reasons
$4x-2y=20$ $5x+2y=7$	Since we have the same number with opposite signs in front of the y's, we can just add the equations.
$\begin{array}{l} 4x-2y=20 \\ \underline{5x+2y=7} \\ 9x \qquad =27 \end{array}$	That way the y's cancel and we can easily solve for x.
$\frac{9x}{9}=\frac{27}{9}$ $x=3$	
$5x+2y=7$ $5(3)+2y=7$	Replace the x in either equation to get the other variable. Here we are using the second equation for no particular reason.
$15+2y=7$ $2y=7-15$ $2y=-8$ $\frac{2y}{2}=\frac{-8}{2}=-4$	
$(3,-4)$ or x=3, y= -4	State the answer either way.

Check by replacing the variables in both equations.

$$\begin{array}{ll} 4x-2y=20 & 5x+2y=7 \\ 4(3)-2(-4)=20 & 5(3)+2(-4)=7 \\ 12+8=20 & 15-8=7 \\ 20=20 & 7=7 \end{array}$$

Both equations check.

$(3,-4)$ is the solution.

9. Solve by the addition method: $\begin{array}{l} 5x-3y=11 \\ 2x+4y=-6 \end{array}$

Steps	Reasons
$4\cdot(5x-3y)=(11)\cdot 4$ $3\cdot(2x+4y)=(-6)\cdot 3$	Try to get the same number (but opposite sign) in front of either variable by multiplying both sides of one or both of the equations.
$20x-12y=44$ $6x+12y=-18$	Here we are going to get rid of y's. Note that with the -12y and +12y we really have gotten the least common multiple of the -3 and +4 with opposite signs.
$\begin{array}{l} 20x-12y=44 \\ \underline{6x-12y=-18} \\ 26x \qquad = 26 \end{array}$	Add the two equations to get rid of one of the variables.
$\frac{26x}{26}=\frac{26}{26}$	Solve for the remaining variable.
$x=1$	
$5x-3y=11$ $5(1)-3y=11$ $-3y=11-5$ $-3y=6$ $y=-2$	Replace the number for the variable in either equation to get the other variable. Here we are using the first equation for no particular reason.
$(1,-2)$ or x=1, y= -2	State the answer either way.

Check by replacing the variables in both equations.

$5x-3y=11$	$2x+4y=-6$	Both equations check.
$5(1)-3(-2)=11$	$2(1)+4(-2)=-6$	$(1,-2)$ is the solution.
$5+6=11$	$2+(-8)=-6$	
$11=11$	$-6=-6$	

10. Solve by the addition method: $\begin{matrix} 3x - 6y = 12 \\ 2x - 4y = 8 \end{matrix}$

Steps	Reasons
$2(3x-6y)=(12)2$ $-3(2x-4y)=(8)(-3)$	Try to get the same number (but opposite sign) in front of either variable by multiplying both sides of one or both of the equations. Notice that we get the least common multiple of the numbers in front of the x's with opposite signs.
$6x-12y=24$ $-6x+12y=-24$	
$\begin{array}{r} 6x-12y=24 \\ \underline{-6x-12y=-24} \\ 0 \;=\; 0 \end{array}$	Add the two equations to get rid of one of the variables. As it turns out, we got rid of both variables.
There is a line of solutions. or There are an infinites number of solutions.	State your answer either way. Since zero always equals zero, the two equations represent the same line.

3 types of answers:

1. The solution is a point: **Independent systems of equations** have a one-point solution. If we graph both lines, they intersect at one point.

2. No solutions. We get a false equality. 8 = 1 meant there were no solutions. If we graph both lines, they are parallel and never touch. **(Inconsistent system of equations)**

3. An infinite number of solutions or a line of solutions. If we graph both lines, they are the same line. This occurs if we are left with two numbers that really are equal. Above we ended up with 0=0 for instance, and we can say there are an infinite number of solutions or a line of solutions. **(Dependent system of equations)**

Anytime we learn to solve a new type of equation, we are able to solve new types of application problems. There are some hints that we may be using systems of equations:
1. There are two sets of information.
2. We are looking for two unknowns

Examples:

11. Three cans of Coca Cola and one can of Mountain Dew contain 163 grams of sugar. Two cans of Coca Cola and four cans of Mountain Dew contain 262 grams of sugar. How many grams of sugar are there in each?

Let x = amount of sugar in Coca Cola and y = amount of sugar in Mountain Dew

3 cans of Coca Cola and 1 can of Mountain Dew contain 163 grams of sugar
3 x + 1 y = 163

2 cans of Coca Cola and 4 cans of Mountain Dew contain 262 grams of sugar
2 x + 4 y = 262

Solve with either method:

$$3x+y=163$$
$$2x+4y=262$$

Steps	Reasons
$3x+y=163$ $2x+4y=262$	Since y in the first equation does not have a number in front, it is appropriate to use the substitution method.
$3x+y=163$ $y=163-3x$	Solve the first equation for y by subtracting 3x from both sides.
$2x+4(163-3x)=262$	Substitute for y in the other equation. Solve:
$2x+652-12x=262$	Distribute the 4.
$-10x=262-652$	Collect like terms and subtract 652 from both sides.
$-10x=-390$	Divide both sides by -10.
$x=39$	Get y by substituting x into one of the equations.
$y=163-3x$ $y=163-3(39)$ $y=46$	There are 39 grams of sugar in the Coca Cola and 46 grams of sugar in the can of Mountain Dew.

12. A shop has a special rate for changing all the tires on motorcycles or cars. During a week where the shop changed the tires on 27 vehicles at the special rate, 84 tires were replaced. How many motorcycles and how many cars had all the tires replaced?

Let x = number of motorcycles?
y = number of cars?

Letting variables stand for what we are trying to find is often a good trick.

“27 vehicles” means the number of motorcycles plus the number of cars equals 27.
x + y = 27

“84 tires were replaced” means:

the number of motorcycle tires plus the number of car tires equals 84.
2x + 4y = 84

Steps	Reasons
Solve: $\begin{array}{l} x+y=27 \\ 2x+4y=84 \end{array}$	Here we are solving by the addition method even though the substitution method would work as well.
$-2(x+y)=-2(27)$ $2x+4y=84$	Multiply the first equation by -2 so that we have -2x and 2x. Then adding the equation will get rid of the x’s.
$\begin{array}{r} -2x+-2y=-54 \\ 2x+\ 4y=\ \ 84 \\ \hline 2y=30 \end{array}$	Solve for y.
$y=15$	
$x+y=27$ $x+15=27$ $x=12$	Get x by replacing y in the first equation.

The shop changed the tires for 12 motorcycles and 15 cars at the special rate.

13. A broker purchased two bonds for a total of $250,000. One bond earns 7% simple annual interest, and the second one earns 8% simple annual interest. If the total annual interest from the two bonds is $18,500, what was the purchase price of each bond?

Solution
Use a chart to organize the information
Principal = amount of money invested.
Rate = annual simple interest
Interest = dollar amount that was invested
I = Prt is the formula for simple interest. We will see this in finance.

	Principal	Rate	Interest (principal X rate)
Amount at 8%	X +	.08	.08x +
Amount at 7%	Y =	.07	.07y =
	250,000		18,500

Since the total amount is $250,000, we can add each amount to get the total..
Since the interest adds up to $18,500 we get the equation we are need to solve.

Steps	Reasons
$x+y=250{,}000$ $.08x+.07y=18{,}500$	Here we are using the substitution method even though the addition method would work well.
$y=250{,}000-x$ $.08x+.07(250{,}000-x)=18{,}500$	Solve the first equation for y and substitute for y in the second equation.
$.08x+17{,}500-.07x=18{,}500$ $\quad -17{,}500 \qquad\qquad -17{,}500$	Solve the resulting equation.
$.01x=1000$ $x=100{,}000$	
$y=250{,}000-x$ $y=250{,}000-100{,}000=150{,}000$	Get y by substituting for x in one of the equations and evaluating.

$100,000 is invested at 8% and $150,000 is invested at 7%.

14. A restaurant purchased 5 kg of tomatoes and 16 kg of potatoes for a total of \$55.00. A second purchase, at the same prices, included 3 kg of tomatoes and 8 kg of potatoes for a total of \$29.00. Find the cost per kilogram for the tomatoes and the potatoes.

Make two charts: one for each purchase.

First Purchase:

	Amount	Unit cost	Total
Tomatoes	5	T	5T +
Potatoes	16	P	16P =
Total			55

Second Purchase:

	Amount	Unit cost	Total
Tomatoes	3	T	3T +
Potatoes	8	P	8P =
Total			29

Solve the system of equations:
$$\begin{aligned} 5T + 16P &= 55 \\ 3T + 8P &= 29 \end{aligned}$$

Steps	Reasons
$5T+16P=55$ $3T+8P=29$	The two charts yield two equations if you look at the total price for each purchase.
$5T+16P=55$ $-2(3T+8P)=(29)(-2)$	Here we are using the addition method.
$5T+16P=55$ $\underline{-6T-16P=-58}$	The P's cancel and we solve for T.
$-T=-3$	
$T=3$	
$3T+8P=29$ $3(3)+8P=29$	Substitute T = 3 in either equation and then solve for P
$9+8P=29$	
$8P=20$	
$P=2.5$	

The tomatoes cost $3 per kg and the potatoes cost $2.50 per kg.

Exercises

1. Is $(3,1)$ a solution to
   $5x + 2y = 17$
   $3x - 2y = 7$?

2. Is $(-2,4)$ a solution to
   $3x - 2y = -14$
   $5x + y = 6$?

3. Is $(4,-1)$ a solution to
   $2x + 5y = 3$
   $4x - y = 10$?

4. Is $(3,4)$ a solution to
   $3x + 2y = 17$
   $5x - 3y = 7$?

Solve the system of equations by graphing:

5. $x + y = 4$
   $x - y = 2$

6. $2x - 3y = 6$
   $x + y = 3$

7. $y = 2x + 1$
   $y = 2x - 1$

8. $y = \frac{1}{3}x - 4$
   $y = \frac{1}{3}x + 2$

Solve by substitution:

9. $y = 2x + 3$
   $x + y = 6$

10. $y = 2x - 7$
    $x + 2y = 16$

11. $x = 3y - 4$
$2x + 3y = 1$

12. $3x - y = 12$
$x + 2y = 11$

13. $x + 2y = 8$
$2x - y = 1$

14. $3x + y = 11$
$2x - 3y = 22$

15. $x + 3y = 16$
$4x - 2y = -20$

16. $3x - y = -3$
$4x - 5y = 7$

17. $3x - 4y = -7$
$x - 4y = 3$

18. $3x - 2y = 3$
$-5x + y = 9$

Solve by the addition method:

19. $2x - y = 5$
$2x + y = 7$

20. $x + 3y = 14$
$-x + y = 2$

21. $2x + 4y = 2$
$-2x + 3y = 12$

22. $2x - 3y = 17$
$4x + 3y = 7$

23. $5x - 2y = -3$
$3x + y = 7$

24. $3x + y = -2$
    $7x + 2y = -3$

25. $2x - 3y = 7$
    $4x + 2y = 6$

26. $3x - 2y = 23$
    $x - 4y = 21$

27. $3x - 4y = 5$
    $2x + 3y = -8$

28. $5x + 3y = 7$
    $3x + 4y = 13$

29. $3x + 5y = 6$
    $2x - 3y = 23$

30. $4x + 7y = 11$
    $5x + 6y = 22$

31. $5x + 2y = -11$
    $3x - 7y = -23$

32. $-5x - 3y = -1$
    $6x + 5y = -10$

Solve the following by using an appropriate system of equations:

33. Two cans of cola and one can of root beer contain 149 grams of sugar. Three cans of cola and four cans of root beer contain 336 grams of sugar. How many grams of sugar are there in each?

34. Four cans of Fanta Orange and five cans of Sprite contain 403 grams of sugar. One can of Fanta Orange and two cans of Sprite contain 130 grams of sugar. How many grams of sugar are there in each?

35. A shop has a special rate for changing all the tires on motorcycles or cars. During a week where the shop changed the tires on 38 vehicles at the special rate, 110 tires were replaced. How many motorcycles and how many cars had all the tires replaced?

36. A shop has a special rate for changing all the tires on motorcycles or cars. During a week where the shop changed the tires on 55 vehicles at the special rate, 196 tires were replaced. How many motorcycles and how many cars had all the tires replaced?

37. A broker purchased two bonds for a total of $150,000. One bond earns 4% simple annual interest, and the second one earns 2% simple annual interest. If the total annual interest from the two bonds is $5,400, what was the purchase price of each bond?

38. A broker purchased two bonds for a total of $100,000. One bond earns 3% simple annual interest, and the second one earns 5% simple annual interest. If the total annual interest from the two bonds is $4,200, what was the purchase price of each bond?

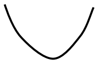
39. At a food stand outside of a European train station it is possible to buy hot dogs and beer. To purchase three hot dogs and two beers it costs 8.25 euro. For two hot dogs and three beers it costs 8.00 euro. How much does the stand charge for a hot dog? How much does it charge for a beer?

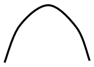
40. At Yankee Stadium in New York City, it costs $21 to purchase three beers and two hot dogs. To purchase two beers and four hot dogs, it costs $30. How much does it cost for a hot dog at Yankee Stadium? How much does it cost for a beer?

Quadratic functions have the form:

$$f(x) = ax^2 + bx + c \text{ where } a,b,c \text{ are real numbers}$$

The graphs of quadratic functions are called parabolas. Their direction is determined by a :

$a > 0$ the shape is up 

$a < 0$ the shape is down 

The lowest or highest point of these parabolas is called the vertex. We can find the first coordinate of the vertex by:

$$x = \frac{-b}{2a} \text{ for } f(x) = ax^2 + bx + c \text{ (the general form of the function)}$$

Thinking of a little piece of the quadratic formula helps us remember the formula. There is a vertical line of symmetry running through the vertex.

As with lines we find:
x-intercepts by letting y = 0.
y-intercept by letting x = 0.

Examples:

1. For the function $f(x) = x^2 - 4x + 3$
    - a. Does the parabola go up or down?
    - b. Find the vertex and state the maximum or minimum.
    - c. Find the x-intercepts.
    - d. Find the y-intercept.
    - e. Graph the function.

Steps	Reasons
$f(x) = x^2 - 4x + 3$	
$a = 1, b = -4, c = 3$	
a. Since a>0, the shape is  The parabola goes up.	Examine the coefficient in front of x^2. Since there is no number in front of the x^2, the coefficient is 1 (think of $x^2 = 1x^2$).
b. Vertex	The x-coordinate of the vertex is $x = \frac{-b}{2a}$.
$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$	Replace a,b with the values from the function.
$f(2) = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$	Find the second coordinate of the vertex by evaluating the function for the x-value.
$(2,-1)$ is the vertex	

c. x-intercepts

$f(x) = x^2 - 4x + 3$

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x - 3 = 0 \quad or \quad x - 1 = 0$

$x = 3 \qquad\quad x = 1$

The x-intercepts are the points $(3,0)$ and $(1,0)$.

To find the x-intercepts replace f(x) or y with zero and solve for x.

The easiest way to solve is by factoring. The factors of 3 that add up to -4 are -3 and -1: (-3)(-1)=3 and -3+(-1)=-4.

There are two x-intercepts. Remember the second coordinate f(x) is 0 from the start.

d. y-intercept

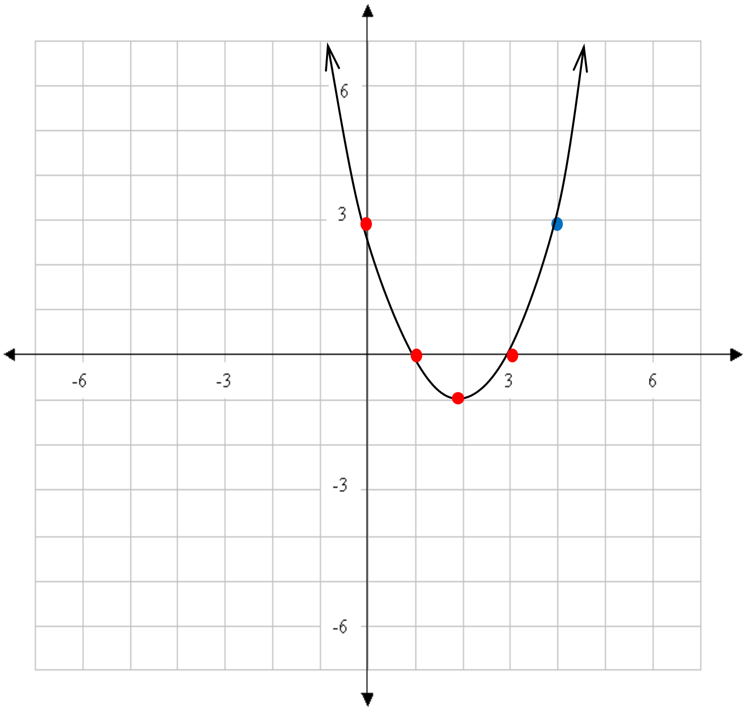
$f(0) = 0^2 - 4(0) + 3$

$= 3$

The y-intercept is $(0,3)$.

To find the y-intercept let x=0. Evaluate the function for f(0).

e. Graph the points and draw the shape of a parabola. If there are not enough points to draw a parabola, then pick a few more x-values to get more points.



Graph the vertex, x-intercepts, and y-intercept (red dots). The blue dot makes sense because parabolas are symmetric.

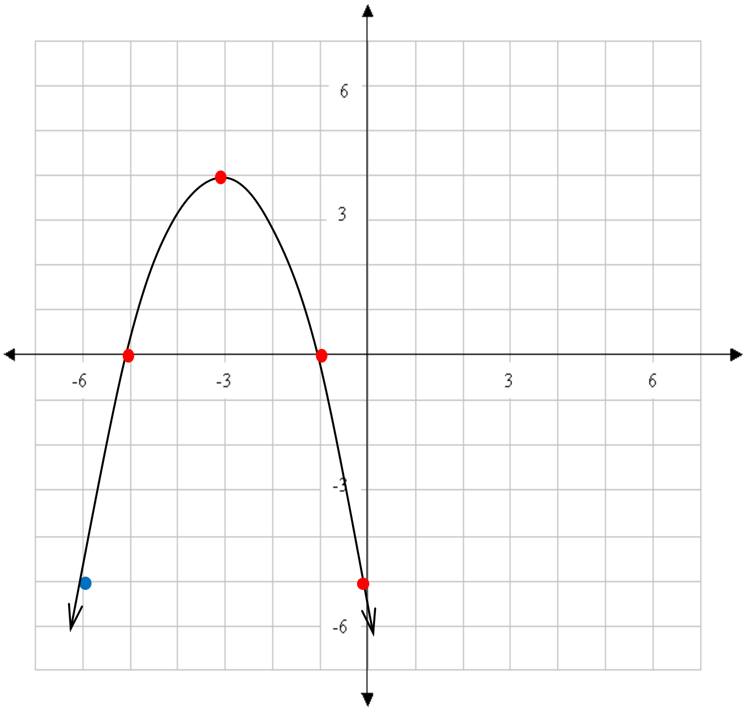
Notice that the vertex (2,-1) is the lowest point on the graph. No matter what x-value is put into the function the minimum value for f(x) is -1.

2. For the function $f(x)=-x^2-6x-5$

   a. Does the parabola go up or down?
   b. Find the vertex and state the maximum or minimum.
   c. Find the x-intercepts.
   d. Find the y-intercept.
   e. Graph the function

<u>Steps</u>	<u>Reasons</u>
$f(x)=-x^2-6x-5$ $a=-1,b=-6,c=-5$	Identify the coefficients (a,b,c) for the quadratic function $f(x)=ax^2+bx+c$
a. Since a<0, the shape is ∩ The parabola goes down.	Examine the coefficient in front of x^2. It is -1 because $-x^2=-1x^2$.
b. Vertex $x=\frac{-b}{2a}=\frac{-(-6)}{2(-1)}=\frac{6}{-2}=-3$	The x-coordinate of the vertex is $x=\frac{-b}{2a}$.
$f(-3)=-(-3)^2-6(-3)-5=-9+18-5=4$ $(-3,4)$ is the vertex.	Find the second coordinate of the vertex by evaluating the function for the x-value.
c. x-intercepts $f(x)=-x^2-6x-5$	To find the x-intercepts replace f(x) or y with zero and solve for x.
$0=-x^2-6x-5$ $-1(0)=-1(-x^2-6x-5)$ $0=x^2+6x+5$	Before trying to solve by factoring it is very helpful to multiply both sides by -1. Making sure that the x^2 term is positive makes the factoring much easier.
$0=(x+5)(x+1)$	The factors of 5 that add up to 6 are 5 and 1. (5)(1)=5 and 5+1=6.
$x+5=0$ *or* $x+1=0$ $x=-5$ $\qquad x=-1$	
The x-intercepts are the points $(-5,0)$ and $(-1,0)$.	There are two x-intercepts. Remember the second coordinate f(x) is 0.
d. y-intercept $f(0)=-0^2-6(0)-5=-5$ The y-intercept is $(0,-5)$.	To find the y-intercept let x=0. Evaluate the function for f(0).

e. Graph the points and draw the shape of a parabola. If there are not enough points to draw a parabola, then pick a few more x-values to get more points.



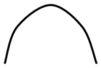
Graph the vertex, x-intercepts, and y-intercept (red dots). The blue dot makes sense because parabolas are symmetric.

Notice that the vertex (-3,4) is the highest point on the graph. That means that regardless of the x-values the maximum value for f(x) is 4.

Remember the shape of the parabola is determined by the a :

$a > 0$ the shape is up 

Here there is a lowest point or minimum. The minimum of the quadratic function is the second coordinate of the vertex when $a > 0$

$a < 0$ the shape is down 

Here there is a highest point or maximum. The maximum of the quadratic function is the second coordinate of the vertex when $a < 0$.

Examples:

3. Find the maximum or minimum of the function $f(x)=2x^2-12x+30$

Steps	Reasons
$f(x)=2x^2-12x+30$	Find the vertex:
$x=\frac{-b}{2a}$ $x=\frac{-(-12)}{2(2)}=\frac{12}{4}=3$	1. Use $x=\frac{-b}{2a}$ to find the x-coordinate of the vertex.
$f(3)=2(3)^2-12(3)+30$ $=2\cdot 9-36+30$ $=12$	2. Replace x with 3 to get the second coordinate.
The minimum is 12	Because $a>0$ the shape is up  So, there is a lowest point or minimum. The minimum of the quadratic function is the second coordinate of the vertex when $a>0$

We can use this notion of finding maximums and minimums to solve important word problems. In business we want to maximize profit and minimize cost.

4. A manufacturer believes that the profit in dollars, P, the company makes is related to the number of coffee makers produced and sold, x, by the function $P(x)=-0.03x^2+60x-7000$. What is the maximum profit that the manufacturer can expect?

Steps	Reasons
$P(x)=-0.03x^2+60x-7000$	Find the vertex:
$x=\frac{-b}{2a}=\frac{-60}{2(-0.03)}=\frac{-60}{-0.06}=1000$	Use $x=\frac{-b}{2a}$ to find the x-coordinate of the vertex.
$P(1000)=-0.03(1000)^2+60(1000)-7000$ $=-0.03\cdot 1{,}000{,}000+60{,}000-7000$ $=-30{,}000+60{,}000-7000$ $=23{,}000$	Replace x with 1000 to get the second coordinate.
The maximum profit of $23,000 occurs when 1000 coffee makers are produced and sold.	Write the answer.

Exercises

Answer a-e for the following quadratic functions:

a. Does the parabola go up or down?
b. Find the vertex and state the maximum or minimum.
c. Find the x-intercepts.
d. Find the y-intercept.
e. Graph the function

1. $\mathrm{f(x) = x^2 + 4x + 3}$

2. $\mathrm{f(x) = x^2 + 6x + 5}$

3. $\mathrm{f(x) = x^2 + 2x - 3}$

4. $\mathrm{f(x) = -x^2 + 2x + 3}$

5. $\mathrm{f(x) = -x^2 + 4x - 3}$

6. $\mathrm{f(x) = 2x^2 - 4x - 6}$

7. $\mathrm{f(x) = 2x^2 - 8x + 6}$

8. $\mathrm{f(x) = -2x^2 - 4x + 6}$

9. $\mathrm{f(x) = -2x^2 + 4x + 6}$

10. $\mathrm{f(x) = -2x^2 + 12x - 10}$

Find the maximum or minimum of the function and say whether it is a maximum or minimum.

11. $\mathrm{f(x) = 3x^2 + 18x - 7}$

12. $\mathrm{f(x) = 5x^2 - 20x + 11}$

13. $\mathrm{f(x) = -6x^2 - 24x + 18}$

14. $\mathrm{f(x) = -7x^2 + 28x + 15}$

15. $\mathrm{f(x) = 15x^2 - 120x + 100}$

16. $f(x) = -20x^2 + 2000x - 10{,}000$

17. $f(x) = -100x^2 + 5000x + 20{,}000$

18. $f(x) = 80x^2 - 4800x + 15{,}000$

Solve the following application problems:

19. A company that installs security systems believes that the profit in dollars, P, the company makes is related to the number of security systems installed, x, by the function $P(x) = -0.04x^2 + 1600x - 12{,}000$. What is the maximum profit that the company can expect?

20. A company that rents portable toilets believes that the profit in dollars, P, the company makes is related to the number of portable toilets rented, x, by the function $P(x) = -0.1x^2 + 500x - 18{,}500$. What is the maximum profit that the company can expect?

21. An online clothing company is deciding whether or not to sell a certain type of boot. The company believes that the profit in dollars, P, is related to the number of boots produced and sold, x, by the function $P(x) = -0.4x^2 + 240x - 1500$. What is the maximum profit that the company can expect?

22. An object launched straight up at a speed of 14.7 meters per second has a height, h, in meters of $h(t) = -4.9t^2 + 14.7t$, t seconds after the object is launched. What is the maximum height that the object will reach? Why does t have to be between zero and three? (This problem does not take into account air resistance.)

23. An object launched straight up at a speed of 29.4 meters per second has a height, h, in meters of $h(t) = -4.9t^2 + 29.4t$, t seconds after the object is launched. What is the maximum height that the object will reach? Why does t have to be between zero and six? (This problem does not take into account air resistance.)

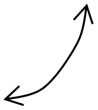
24. A .44 magnum bullet leaves the barrel of the gun at about 392 meters per second. If a .44 magnum bullet is shot straight up in the air, it has a height, h, in meters of $h(t) = -4.9t^2 + 392t$, t seconds after the bullet leaves the barrel. What is the maximum height that the object will reach? (This problem does not take into account air resistance. The actual height of the .44 magnum bullet will be much less than the value we found because of air resistance.)

An exponential function has the form:

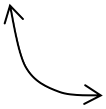
$f(x)=b^x$, b is called the base and b is a positive real number. Also, b cannot be equal to one.

When we graph an exponential function, we get two main types of graphs.

1. $b>1$



2. $0<b<1$



We can draw graphs of $f(x)=b^x$ by picking points until we get the above shapes.

Examples:

1. Graph $f(x)=2^x$

Pick some first coordinates and calculate the second coordinates using the function: $f(x)=2^x$ using -3, -2, -1, 0, 1, 2, 3 will usually be a good start.

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$$f(-2)=2^{-2}=\frac{1}{2^2}=\frac{1}{4}$$
$$f(-1)=2^{-1}=\frac{1}{2^1}=\frac{1}{2}$$
$$f(0)=2^0=1$$
$$f(1)=2^1=2$$
$$f(2)=2^2=4$$
$$f(3)=2^3=8$$

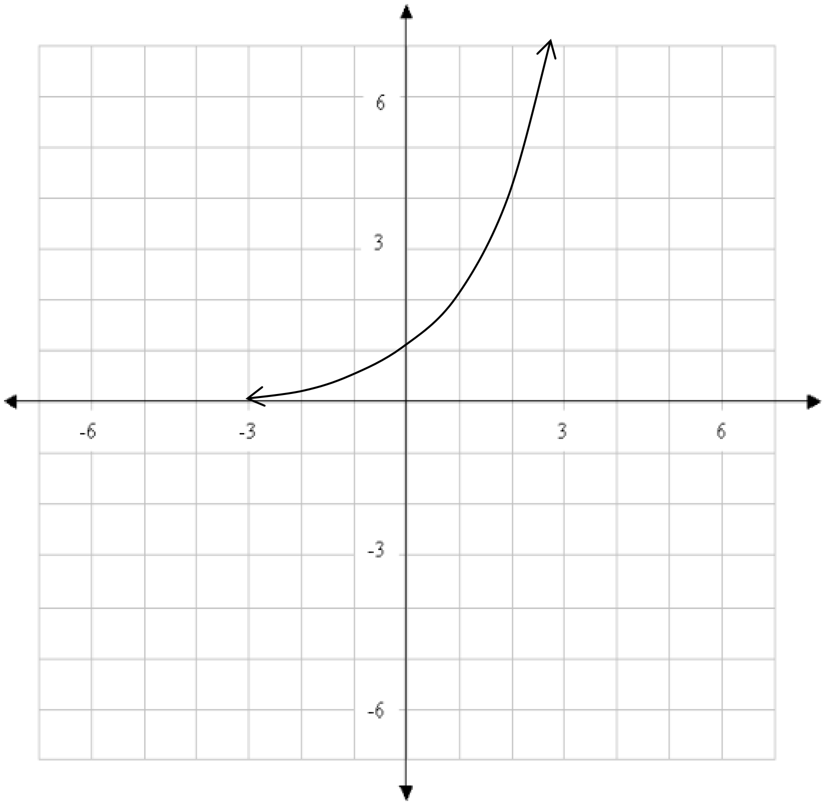
Remember how to evaluate negative exponents:

$$b^{-m}=\frac{1}{b^m}$$

$$b^0=1$$

Note the following:
1. The graph is increasing and has the same shape as $b>1$ from above.
2. On the left the graph gets closer to the x-axis without touching.

$f(x) = 2^x$



2. Graph $f(x) = 3^x$

Pick some first coordinates and calculate the second coordinates using the function: $f(x) = 3^x$

x	$f(x)$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

$$f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$
$$f(-1) = 3^{-1} = \frac{1}{3^1} = \frac{1}{3}$$
$$f(0) = 3^0 = 1$$
$$f(1) = 3^1 = 3$$
$$f(2) = 3^2 = 9$$

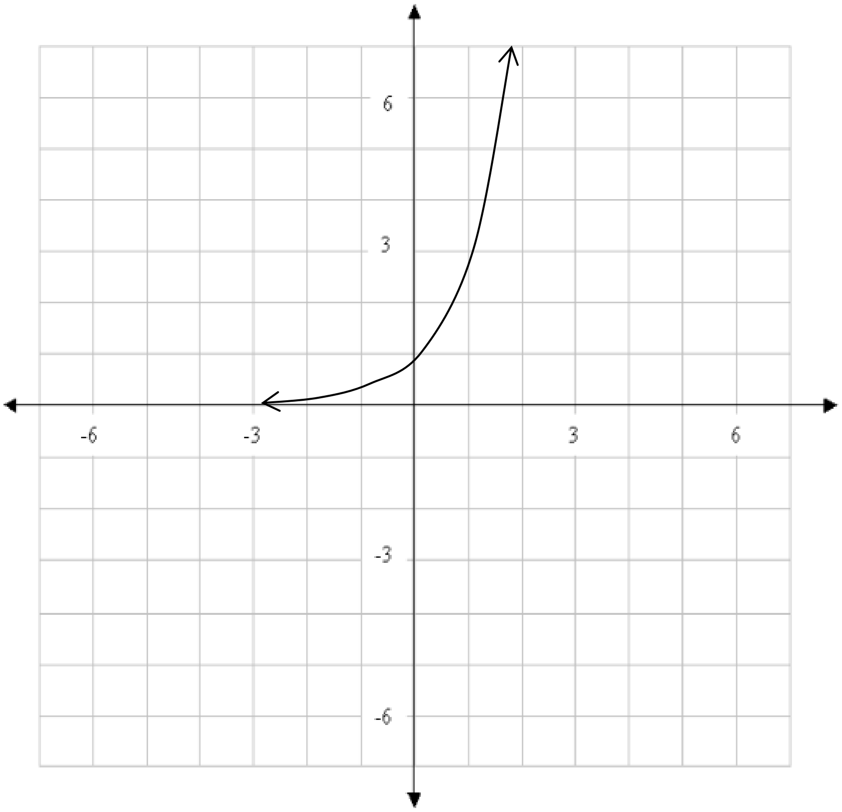
Remember how to evaluate negative exponents:

$$b^{-m} = \frac{1}{b^m}$$
$$b^0 = 1$$

Note the following:

1. The graph is increasing and has the same shape as $b > 1$ from above.
2. On the left the graph gets closer to the x-axis without touching.

$f(x)=3^x$



3. Graph $f(x)=\left(\frac{1}{2}\right)^x$

Pick some first coordinates and calculate the second coordinates:

$f(x)=\left(\frac{1}{2}\right)^x$

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$$f(-3)=\left(\frac{1}{2}\right)^{-3}=2^3=8$$

$$f(-2)=\left(\frac{1}{2}\right)^{-2}=2^2=4$$

$$f(-1)=\left(\frac{1}{2}\right)^{-1}=2^1=2$$

$$f(0)=\left(\frac{1}{2}\right)^{0}=2^0=1$$

$$f(1)=\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$$

$$f(2)=\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$$

To evaluate negative exponents:

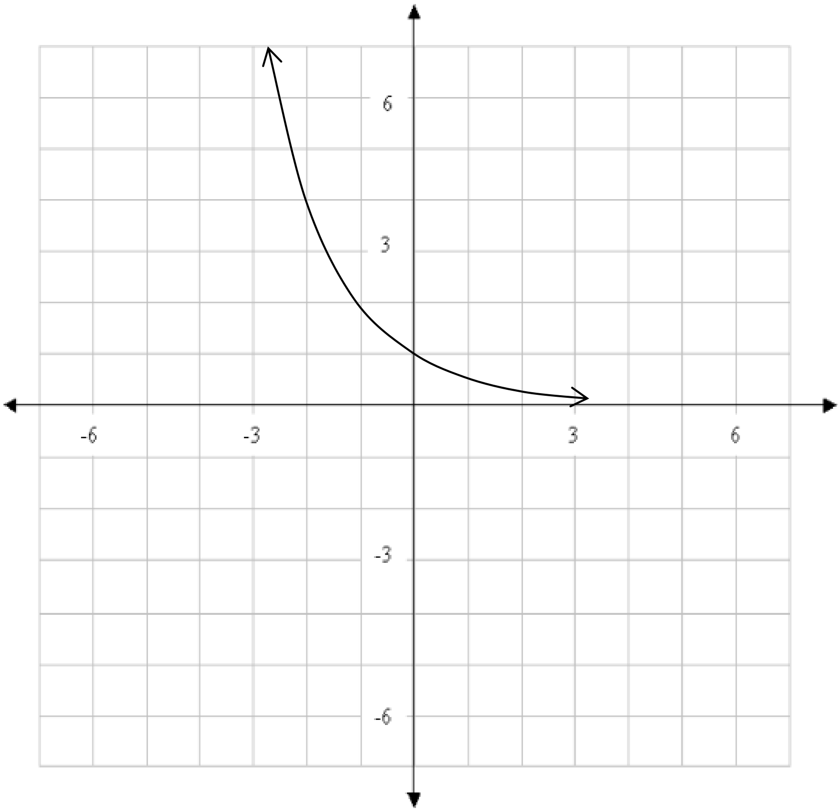
$b^{-m}=\frac{1}{b^m}$ or $\frac{1}{b^{-m}}=b^m$

$b^0=1$

Note the following:

1. The graph is decreasing and has the same shape as $0 < b < 1$ from above.
2. On the right the graph gets closer to the x-axis without touching.

$$f(x) = \left(\frac{1}{2}\right)^x$$



4. Graph: $f(x) = 2^{x-3}$

x	$f(x)$
1	$\frac{1}{4}$
2	$\frac{1}{2}$
3	1
4	2
5	4
6	8

$f(1) = 2^{1-3} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

$f(2) = 2^{2-3} = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

$f(3) = 2^{3-3} = 2^0 = 1$

$f(4) = 2^{4-3} = 2^1 = 2$

$f(5) = 2^{5-3} = 2^2 = 4$

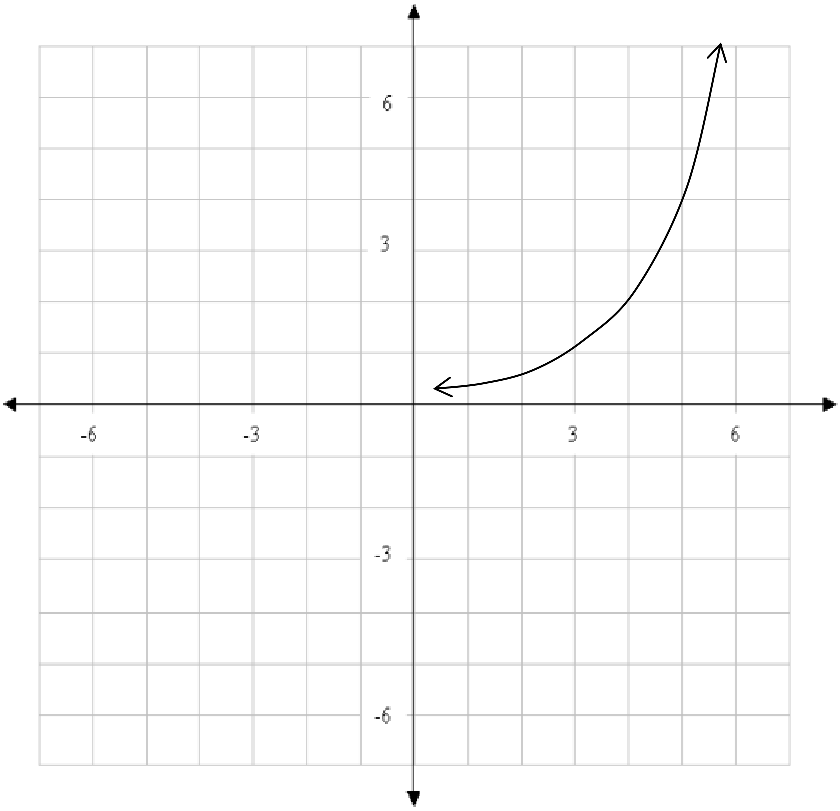
$f(6) = 2^{6-3} = 2^3 = 8$

To evaluate negative exponents:

$b^{-m} = \frac{1}{b^m}$

$b^0 = 1$

$f(x)=2^{x-3}$



Note that this is really the same graph as $f(x)=2^{x}$ moved one unit to the right.

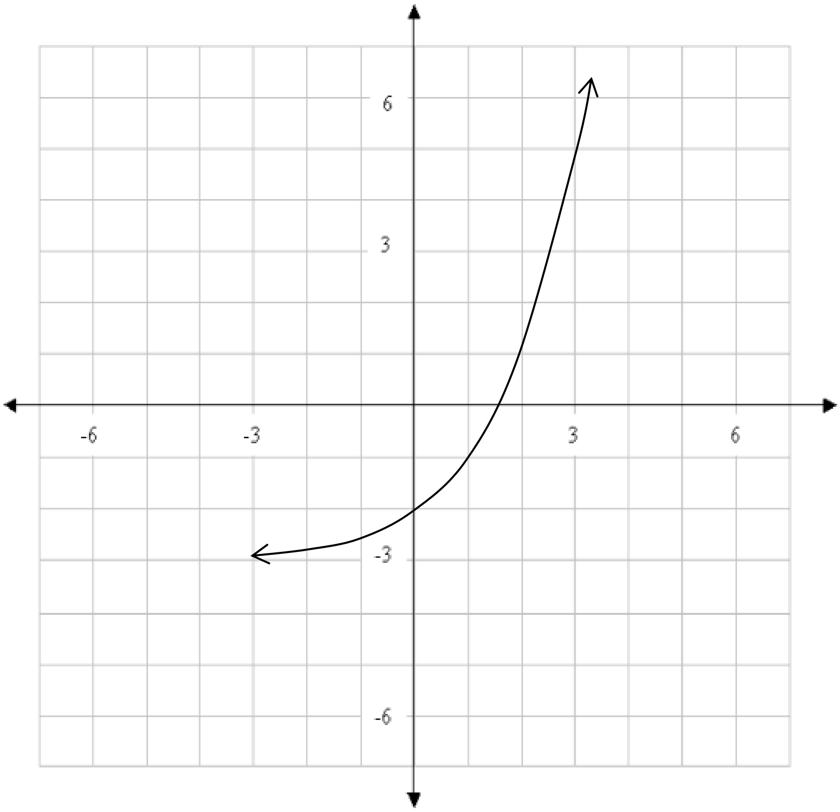
5. Graph: $f(x)=2^{x}-3$

x	$f(x)$
-2	-2.75
-1	-2.5
0	-2
1	-1
2	1
3	5

$$f(-2)=2^{-2}-3=\frac{1}{2^{2}}-3=\frac{1}{4}-3=-2.75$$
$$f(-1)=2^{-1}-3=\frac{1}{2}-3=-2.5$$
$$f(0)=2^{0}-3=1-3=-2$$
$$f(1)=2^{1}-3=2-3=-1$$
$$f(2)=2^{2}-3=4-3=1$$
$$f(3)=2^{3}-3=8-3=5$$

Note that the graph below is really the same graph as $f(x)=2^{x}$ moved down three units.

$f(x) = 2^x - 3$



6. Graph $f(x) = 2^{-x}$

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$f(-3) = 2^{-(-3)} = 2^3 = 8$
$f(-2) = 2^{-(-2)} = 2^2 = 4$
$f(-1) = 2^{-(-1)} = 2^1 = 2$
$f(0) = 2^{-0} = 2^0 = 1$
$f(1) = 2^{-1} = \frac{1}{2}$

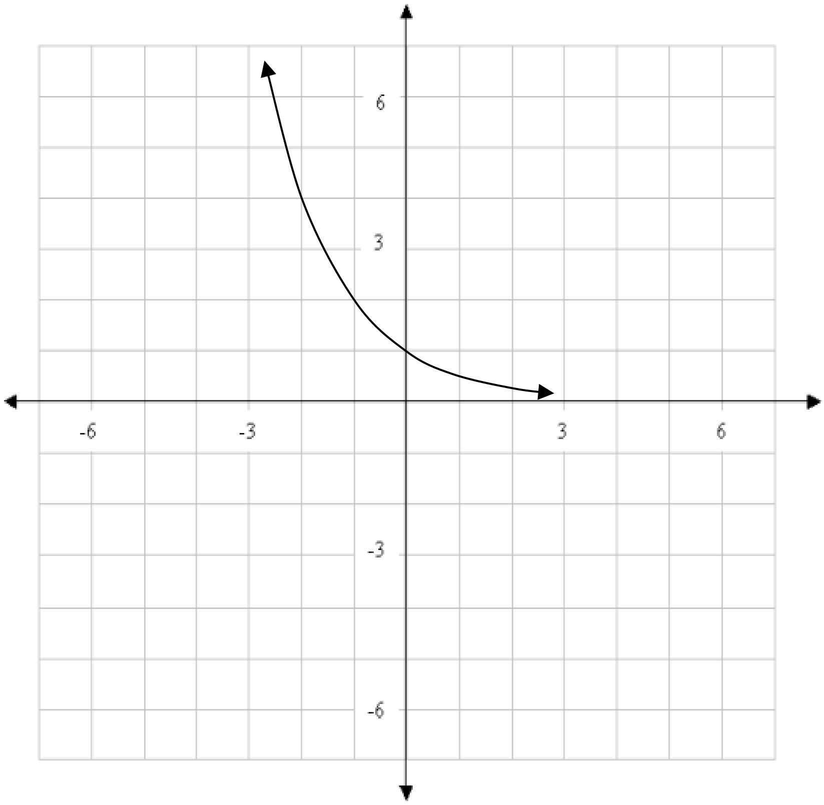
$f(2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

Note the following:

1. The graph is decreasing and has the same shape as $0 < b < 1$ from above.
2. On the right the graph gets closer to the x-axis without touching.

The graphs of $f(x)=\left(\frac{1}{2}\right)^x$ and $f(x)=2^{-x}$ are the same because $2^{-1}=\frac{1}{2}$

$f(x)=2^{-x}$



Calculators:

We can use our calculators to evaluate exponents. Everybody should have a basic scientific calculator with buttons like

- x^y or y^x or $\wedge$
- ln
- log

To find 3^2 on your calculator, usually you have to use one of the following two orders on your calculator. When you get 9, those are the steps that you should follow.

1. Push 3.
2. Push x^y or y^x or $\wedge$. You should have only one.
3. Push 2
4. Push =

or (depending on your calculator)

1. Push 2.
2. Push x^y or y^x or $\wedge$. You should have only one.
3. Push 3
4. Push =

We have a new number "e". It is used in both decay and growth problems. "e" is an irrational number like π. So, we cannot write it as a decimal number, but it is on most calculators. "e" is about 2.71828…

Find e^3 (answer: 20.0855269…)

1. INV, Shift, or 2^{nd}. You should have only one.
2. Push ln.
3. Push 3
4. Push =

or (depending on your calculator)

1. Push 3
2. INV, Shift, or 2^{nd}. You should have only one.
3. Push ln.
4. Push =

Above the "ln" button you usually see the e^x button, which is what we are really using.

Find e^{-4} (answer: 0.0183156…)

1. INV, Shift, or 2^{nd}. You should have only one.
2. Push ln.
3. Push $(-)$ or $\pm$. You should have only one.
4. Push 3
5. Push =

or (depending on your calculator)

1. Push 3
2. Push $(-)$ or $\pm$. You should have only one
3. INV, Shift, or 2^{nd}. You should have only one.
4. Push ln.
5. Push =

Examples:

7. Doubling time is the time it takes for something to double in size, number, or amount. If the world population was about 3 billion in 1960 and 6 billion in 2000 then the doubling time is 40 years. If the world population, $\mathrm{P(t)}$, measured in billions follows the function $\mathrm{P(t)} = 6 \cdot 2^{0.025\,\mathrm{t}}$ where t is the number of years after 2000, what will the world population be in the year 2200?

Solution:

t = 200 because 2200 is 200 years after 2000
Find $P(t)$, by plugging t into the function

$\mathrm{P(t)} = 6 \cdot 2^{0.025\,\mathrm{t}}$
$\mathrm{P(200)} = 6 \cdot 2^{0.025\,(200)} = 192$

In the year 2200 the world's population will be 192 billion if the doubling time remains the same.

8. Carbon-14 is found in all living organisms. When a living organism dies, the carbon-14 begins to slowly decay and become nitrogen-14. We can determine the time since an organism was living by measuring the amount of carbon-14 that remains. The half-life of carbon-14 is about 5730 years, which means that half of the carbon-14 remains 5730 years after a living organism dies.

A living organism that has 50 grams of carbon-14 will have $\mathrm{N(t)} = 50\mathrm{e}^{-0.000121\,\mathrm{t}}$ grams of carbon-14 t years after it dies. How much carbon-14 will remain 10,000 years after the organism dies?

Solution:
t years after the organism dies
$\mathrm{N(t)}$ is the grams of carbon-14

For t = 10,000 find $\mathrm{N(t)}$
$\mathrm{N(10{,}000)} = 50\mathrm{e}^{-0.000121\,(10{,}000)} = 50\mathrm{e}^{-1.21} = 14.90986\ldots$

About 14.9 grams of carbon-14 will remain after 10,000 years.

Exercises

Graph the following on graph paper using a straight-edge for the axes. Remember that graph paper can be downloaded from the Internet easily.

1. $\mathrm{f(x) = 3^x}$
2. $\mathrm{f(x) = 4^x}$
3. $\mathrm{f(x)} = \left(\frac{1}{3}\right)^{\mathrm{x}}$
4. $\mathrm{f(x)} = \left(\frac{1}{4}\right)^{\mathrm{x}}$
5. $\mathrm{f(x) = 2^{x+1}}$
6. $\mathrm{f(x) = 2^{x-2}}$
7. $\mathrm{f(x) = 3^{x-3}}$
8. $\mathrm{f(x) = 3^{x+2}}$
9. $\mathrm{f(x) = 2^x - 5}$
10. $\mathrm{f(x) = 2^x + 1}$
11. $\mathrm{f(x) = 3^x - 4}$
12. $\mathrm{f(x) = 3^x + 2}$
13. $\mathrm{f(x) = 3^{-x}}$
14. $\mathrm{f(x) = 4^{-x}}$
15. Doubling time is the time it takes for something to double in size, number, or amount. The number of people with a disease may double over a few months, years, or even decades. In a particular country if the number of people with malaria, $N(t)$, measured in millions follows the function $N(t) = 5 \cdot 2^{0.04t}$ where t is the number of years after 1960, how many people had malaria in 1960? In

1985? How long was the doubling time for the number of people with malaria according to the formula?

16. Doubling time is the time it takes for something to double in size, number, or amount. For instance the number of rabbits in an area may double quickly under ideal circumstances. If the number of rabbits in area, $\mathrm{N(t)}$, follows the function
$\mathrm{N(t)} = 100 \cdot 2^{0.125\,\mathrm{t}}$ where t is the number of weeks after some start date. How many rabbits are there 8 weeks later? How many rabbits are there 16 weeks later? How long was the doubling time for the number of rabbits according to the formula?

For the next two problems we will consider carbon-14, which is found in all living organisms. When a living organism dies, the carbon-14 begins to slowly decay and become nitrogen-14. We can determine the time since an organism was living by measuring the amount of carbon-14 that remains. The half-life of carbon-14 is about 5730 years, which means that half of the carbon-14 remains 5730 years after a living organism dies.

17. A living organism that has 50 grams of carbon-14 will have $\mathrm{N(t)} = 50\mathrm{e}^{-0.000121 \cdot \mathrm{t}}$ grams of carbon-14 t years after it dies. How much carbon-14 will remain 10,000 years after the organism dies?

18. A living organism that has 80 grams of carbon-14 will have $\mathrm{N(t)} = 80\mathrm{e}^{-0.000121 \cdot \mathrm{t}}$ grams of carbon-14 t years after it dies. How much carbon-14 will remain 7000 years after the organism dies?

Definition of a logarithm:

$y = \log_b x$ means $x = b^y$

x > 0, b > 0, b≠1

Note that we just switched the x and y in the exponential relation. We are able to go back and forth writing either the logarithmic form or exponential form.

Examples:

1. Write $81 = 3^4$ in logarithmic form.

Steps	Reasons
$81 = 3^4$	$y = \log_b x$ means $x = b^y$ I focus on:
$= \log$	1. The bases are the same.
$= \log_3$	2. The number by itself for the logarithm is the exponent in exponential form.
$4 = \log_3$	3. The last number goes into the only remaining position.
$4 = \log_3 81$	You can do this showing only the last step.

2. Write $2^{-3} = \frac{1}{8}$ in logarithmic form.

Steps	Reasons
$2^{-3} = \frac{1}{8}$	$y = \log_b x$ means $x = b^y$ I focus on:
$= \log$	The bases are the same.
$= \log_2$	The exponent in exponential form is the number by itself for the logarithm.
$-3 = \log_2$	The last number goes into the only remaining position.
$-3 = \log_2 \frac{1}{8}$	You can do this showing only the last step.

3. Write the $\log_5 \frac{1}{25} = -2$ in exponential form

Steps	Reasons
$\log_5 \frac{1}{25} = -2$	The bases are the same.
$= 5$	The number by itself for the logarithm is the exponent in exponential form.
$= 5^{-2}$	The last number goes into the only remaining position.
$\frac{1}{25} = 5^{-2}$	You can do this showing only the last step.

There are some special logarithms where the base is not written, but understood. These logarithms are on scientific calculators.

The common logarithm is base 10:
$\log x = \log_{10} x$ If no base is written, then it is base 10.

The natural logarithm is base e:
$\ln x = \log_e x$ If ln is written, then it is base e. $e \approx 2.7.1828...$ Since e is an irrational number, we cannot write it out as a decimal number without rounding off. The number e comes by looking at continuous growth and decay.

Steps for evaluating logarithms:

1. Let y = the logarithm
2. Rewrite as an exponential relation.
3. Determine what y should be.
4. The y will be the answer.

Examples:
4. Evaluate $\log 1000$.

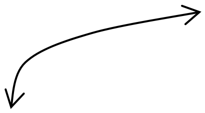
Steps	Reasons
$\log 1000$	Let y = the logarithm. If no base is written, then it is base 10.
$y = \log 1000$	
$y = \log_{10} 1000$	Rewrite as an exponential relation using
$1000 = 10^y$	$y = \log_b x$ means $x = b^y$
$y = 3$	Determine what y should be. The y is the answer.
$\log 1000 = 3$	Write the answer.

5. Evaluate $\ln e^4$.

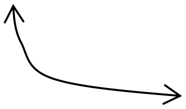
Steps	Reasons
$\ln e^4$	
	Let y = the logarithm.
$y = \ln e^4$	
	If ln is written, then it is base e.
$y = \log_e e^4$	
$e^4 = e^y$	Rewrite as an exponential relation using $y = \log_b x$ means $x = b^y$
$y = 4$	Determine what y should be. The y is the answer.
$\ln e^4 = 4$	Write the answer.

All logarithmic functions have one of the following two shapes depending on whether the base is greater than one or between zero and one.

$y = \log_b x$ for b > 1 has the shape:



$y = \log_b x$ for 0 < b < 1 has the shape:



Steps for graphing logarithmic functions:

1. Replace $f(x)$ with y
2. Rewrite the logarithmic equation as an exponential equation using the definition: $y = \log_b x$ means $x = b^y$
3. Pick some y values (-2,-1,0,1,2 for instance) and calculate the x coordinate.
4. Graph the points and draw the graph as above. More points may be needed if the shape is not clear.

Examples:

6. Graph: $f(x) = \log_3 x$

1. Replace $f(x)$ with y

$y = \log_3 x$

2. Rewrite the logarithmic equation as an exponential equation using the definition: $y = \log_b x$ means $x = b^y$

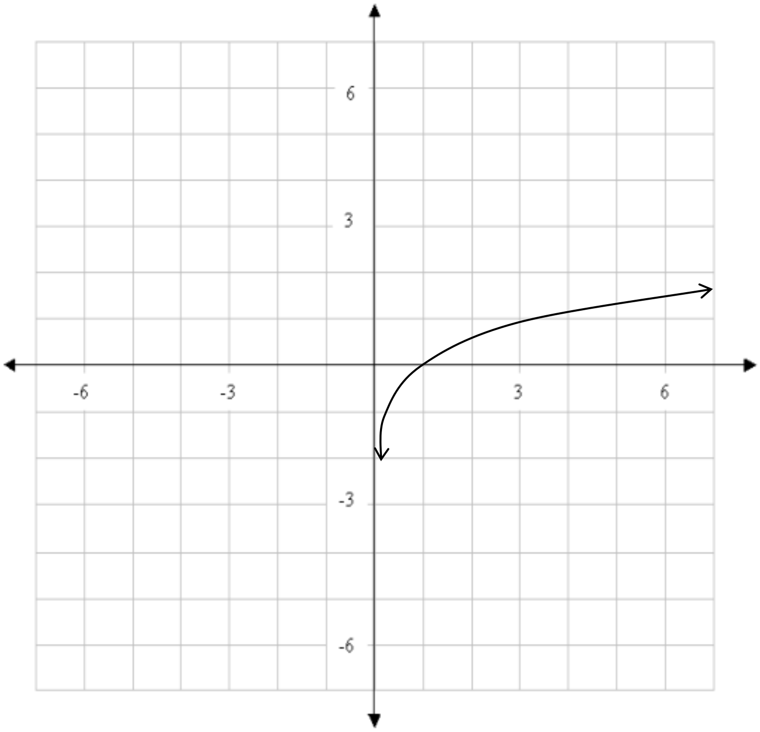
$x = 3^y$

3. Pick some y values (-2,-1,0,1,2 for instance) and calculate the x coordinate.

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$
$$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$$
$$3^0 = 1$$
$$3^1 = 3$$
$$3^2 = 9$$

Pick y-values and calculate x-values. Be careful to keep the x's in the first column and y's in the second.



Notice that the graph approaches a vertical line on the left.

7. Graph: $f(x)=\log_3(x+2)$

$y=\log_3(x+2)$

1. Replace $f(x)$ with y
2. Rewrite the logarithmic equation as an exponential equation using the definition: $y=\log_b x$ means $x=b^y$
3. Pick some y values (-2,-1,0,1,2 for instance) and calculate the x coordinate.

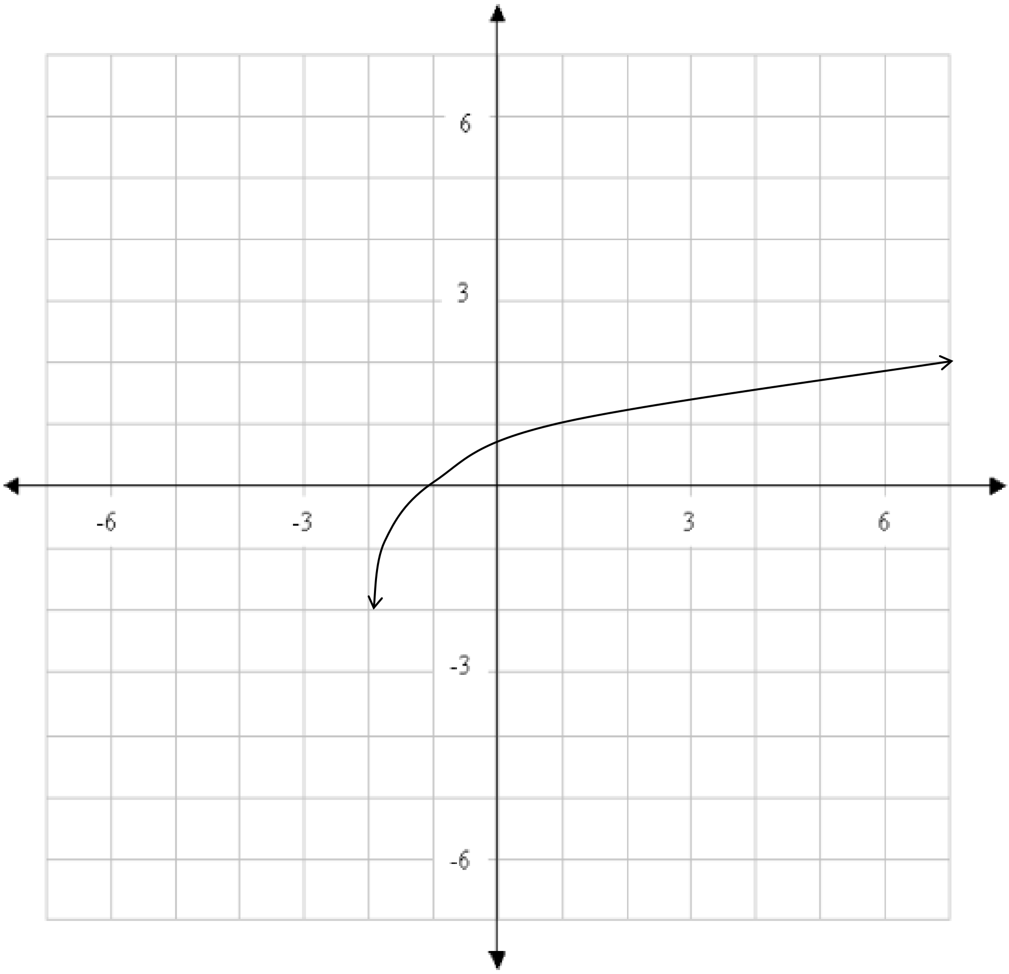
$x+2=3^y$

$x=3^y-2$

x	y
$-1\frac{8}{9}$	-2
$-1\frac{2}{3}$	-1
-1	0
1	1
7	2

$$3^{-2}-2=\frac{1}{3^2}-2=\frac{1}{9}-2=-1\tfrac{8}{9}$$
$$3^{-1}-2=\frac{1}{3^1}-2=\frac{1}{3}-2=-1\tfrac{2}{3}$$
$$3^0-2=1-2=-1$$
$$3^1-2=3-2=1$$
$$3^2-2=9-2=7$$

Pick y-values and calculate x-values. Be careful to keep the x's in the first column and y's in the second.



Notice that the graph approaches a vertical line on the left.

8. Graph: $f(x)=\log_{\frac{1}{2}} x$

1. Replace $f(x)$ with y

$y=\log_{\frac{1}{2}} x$

2. Rewrite the logarithmic equation as an exponential equation using the definition: $y=\log_b x$

$x=\left(\frac{1}{2}\right)^y$

3. Pick some y values (-3,-2,-1,0,1,2, coordinate.

Pick y-values and calculate x-values. Be careful to keep the x's in the first column and y's in the second.

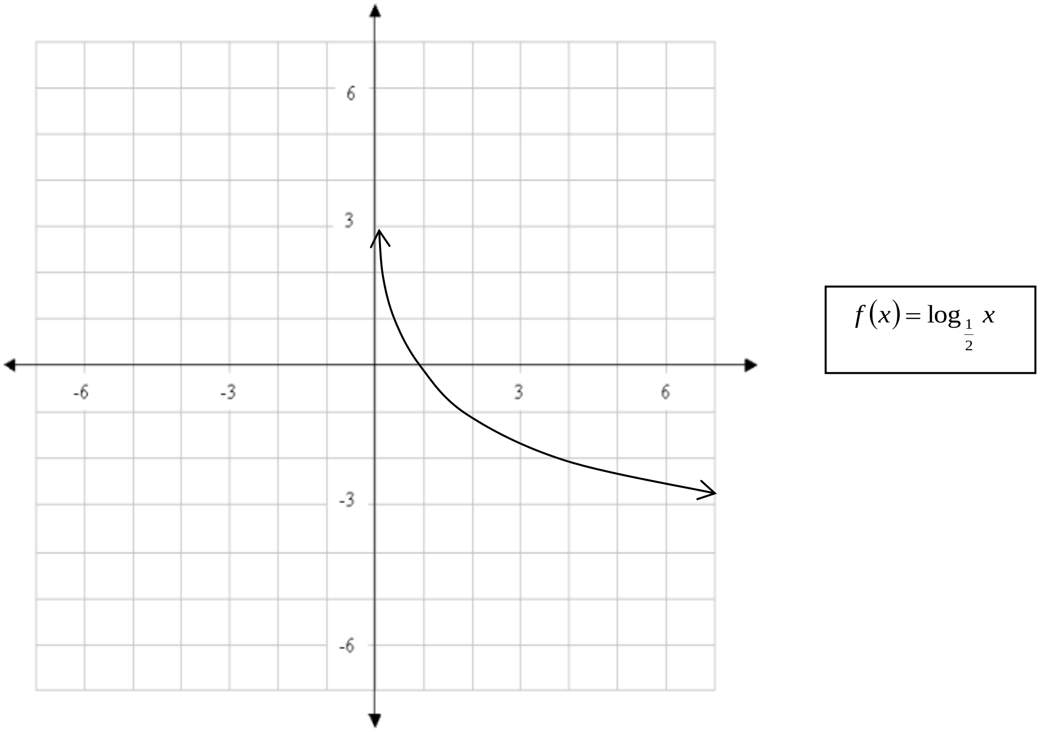
x	y
$\frac{1}{8}$	3
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3

$\left(\frac{1}{2}\right)^{-3}=2^3=8$ $\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$

$\left(\frac{1}{2}\right)^{-2}=2^2=4$ $\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$

$\left(\frac{1}{2}\right)^{-1}=2^1=2$ $\left(\frac{1}{2}\right)^{3}=\frac{1}{8}$

$\left(\frac{1}{2}\right)^{0}=1$



9. Advertising dollars are well spent up to a point. For instance a well-known company may decide to spend another ten million dollars, but it is unlikely that they will reach people that have not heard of their product. Suppose the function $\mathrm{f(x)} = 10{,}000 \cdot \log(\mathrm{x})$ models the number of people that will hear of a new product for x equal to the amount spent on advertising in dollars. How many people will have heard of the new product after $500,000 is spent on advertising? After $1,000,000 is spent? Is it worth spending the extra money on advertising?

Steps	Reasons
$\mathrm{f(x)} = 10{,}000 \cdot \log(x)$	Plug x = 500,000 into the function.
$\mathrm{f}(500{,}000) = 10{,}000 \cdot \log(500{,}000)$	Use a scientific calculator with a “log”
$\mathrm{f}(500{,}000) = 56{,}989.700\ldots$	button, which is log base 10.

About 57,000 people will have heard of the new product after $500,000 is spent on advertising.

$\mathrm{f(x)} = 10{,}000 \cdot \log(x)$	Plug x = 1,000,000 into the function.
$\mathrm{f}(1{,}000{,}000) = 10{,}000 \cdot \log(1{,}000{,}000)$	Use a scientific calculator with a “log”
$\mathrm{f}(1{,}000{,}000) = 60{,}000$	button, which is log base 10.

About 60,000 people will have heard of the new product after $1,000,000 is spent on advertising.

Since only 3000 more people will hear about the new product after spending another $500,000, it is probably not worth spending the extra half million dollars on advertising.

10. For exponential growth, doubling time is the time that it takes for the amount to double. Doubling time is related to the continuous growth rate by $\ln 2 = \mathrm{k} \cdot \mathrm{t}$ where k is the growth rate and t is the doubling time. If it takes 30 years for prices to double, what is the growth rate, which is also known as inflation?

Steps	Reasons
$\ln 2 = \mathrm{k} \cdot \mathrm{t}$	Write the formula.
$\ln 2 = \mathrm{k} \cdot 30$	Replace t with the 30 year doubling time.
$\frac{\ln 2}{30} = \mathrm{k}$	Solve for k and use a scientific calculator to find k.
$\mathrm{k} = 0.0231\ldots \text{ or } 2.3\%$	

If the doubling time is 30 years, then inflation (or growth rate) is 2.3%.

Exercises

Write in logarithmic form:

1. $100 = 10^2$

2. $10{,}000 = 10^4$

3. $8 = 2^3$

4. $1 = 7^0$

5. $\frac{1}{9} = 3^{-2}$

6. $\frac{1}{25} = 5^{-2}$

Write in exponential form:

7. $2 = \log_4 16$

8. $3 = \log_5 125$

9. $0 = \log_5 1$

10. $-4 = \log_{10} 0.0001$

11. $-2 = \log_7 \frac{1}{49}$

12. $-4 = \log_3 \frac{1}{81}$

Graph the following:

13. $y = \log_2 x$

14. $y = \log_4 x$

15. $y = \log_2(x+3)$

16. $y = \log_2(x-1)$

17. $y = \log_{\frac{1}{3}} x$

18. $y = \log_{\frac{1}{4}} x$

19. $y = \log_3(x + 1)$

20. $y = \log_3(x - 2)$

Solve the following:

21. Advertising dollars are well spent up to a point. For instance a well-known company may decide to spend another ten million dollars, but it is unlikely that they will reach people that have not heard of their product. Suppose the function $f(x) = 1000 \cdot \log(x)$ models the number of people that will first hear of a particular new product for x equal to the amount spent on advertising in dollars. How many people will hear of the new product after $40,000 is spent on advertising? After $1,500,000? Is it worth spending the extra money on advertising?

22. Advertising dollars are well spent up to a point. For instance a well-known company may decide to spend another ten million dollars, but it is unlikely that they will reach people that have not heard of their product. The function $f(x) = 100{,}000 \cdot \log(x)$ models the number of people that will hear of a particular new product for x equal to the amount spent on advertising in dollars. How many people will have heard of the new product after $2,500,000 is spent on advertising? After $25,000,000? Is it worth spending the extra money on advertising?

23. For exponential growth, doubling time is the time that it takes for the amount to double. Doubling time is related to the continuous growth rate by $\ln 2 = k \cdot t$ where k is the growth rate and t is the doubling time. If it takes 7 years for prices to double, what is the growth rate, which is also known as inflation?

24. For exponential growth, doubling time is the time that it takes for the amount to double. Doubling time is related to the continuous growth rate by $\ln 2 = k \cdot t$ where k is the growth rate and t is the doubling time. If it takes 30 years for the population to double, what is the growth rate?

6

Students learn about finance as it applies to their daily lives. Two of the most important types of financial decisions for many people involve either buying a house or saving for retirement. Students explore retirement accounts and mortgages. Understanding the quantitative side helps people make better decisions.

Course Outcomes:

- Recognize and apply mathematical concepts to real-world situations
- Efficiently use relevant technology
- Identify and solve problems in finance

6.1 Simple Interest

Simple interest concepts are developed. Students learn to find simple interest and future value. Simple interest rate and principle are found using the future value formula and algebra.

6.2 Compound Interest

Compound interest concepts are developed. Students solve compound interest problems using their calculators and compound interest formulas for present value, future value, and future value for continuous compounding.

6.3 Annuities

Students learn to find periodic payments and future value for annuities. There is a focus on retirement accounts. The advantage of time is explored.

6.4 Mortgages

The different aspects of mortgages are discussed: down payment, points, monthly payments, interest. Students use the formula for monthly payment to better understand the advantages of shorter mortgages as opposed to longer mortgages.

People have been borrowing and lending money for millennia. Whether you borrow money to buy a car or put money in the bank, there will be some type of interest involved. The first type of interest that we will study is called simple interest. The formula for simple interest is

$\mathrm{Int} = \mathrm{Prt}$

$\mathrm{Int} =$ simple interest (a dollar amount)
$\mathrm{P} =$ principal or present value which is the amount borrowed
$\mathrm{r} =$ annual simple interest rate
$\mathrm{t} =$ time measured in years

For simple interest the amount is applied one time at the end of the borrowing period. Notice that when we talk about interest, we are referring to a dollar amount. When we talk about simple interest rate, we are talking about the percent of the principal.

Examples

3. A construction company borrows \$30,000 for 2 years at 5% simple interest. How much interest must the construction company pay for the use of the money?

Steps	Reasons
$\mathrm{Int} = \mathrm{Prt}$	Recognize the type of interest and write down the formula.
$\mathrm{Int} =$???	We are looking for the interest, which will be a dollar amount.
$\mathrm{P} =$ \$30,000	We have the amount borrowed, which is P.
$\mathrm{r} =$ 5% or .05	The simple interest rate is given.
$\mathrm{t} =$ 2 years	The number of years that the money is borrowed.
$\mathrm{Int} = \mathrm{Prt}$	Substitute the numbers into the formula.
$\mathrm{I} = 30{,}000 \cdot .05 \cdot 2$	Use a calculator to find the simple interest I .
$\mathrm{I} = \$3000$	

The construction company must pay \$3000 in simple interest.

4. Find the simple interest for a principal of \$5400 that is borrowed at simple interest rate 3% for a period of 8 months.

Steps	Reasons
$\mathrm{Int} = \mathrm{Prt}$	Recognize the type of interest and write down the formula.
$\mathrm{Int} =$???	We are looking for the interest, which will be a dollar amount.
$\mathrm{P} =$ \$5400	The principal is P.
$\mathrm{R} =$ 3% or .03	The simple interest rate is given.
$\mathrm{t} =$ 8 months	The number of years that the money is borrowed.

Here we have to make an adjustment. The 8 months needs to be converted to years. Many students will quickly divide the 8 months by 12 to get the number of years, which is correct. There is a more organized way.

$$\mathrm{t} = 8\ \mathrm{months} \cdot \frac{1\ \mathrm{year}}{12\ \mathrm{months}} = \frac{8}{12}\ \mathrm{or}\ \frac{2}{3}\ \mathrm{years}$$

The fraction $\frac{1\ \mathrm{year}}{12\ \mathrm{months}}$ is called a unit fraction. It has a value of 1. So, we do not change the value of the time. It is true that $8\ \mathrm{months} = \frac{2}{3}\ \mathrm{year}$. This organized method of changing the units is called dimensional analysis.

$\mathrm{Int} = \mathrm{Prt}$ Substitute the numbers into the formula.

$\mathrm{Int} = 5400 \cdot .03 \cdot \left(\frac{2}{3}\right)$ Use a calculator to find the simple interest Int .

$\mathrm{Int} = \$108$ The simple interest is $108.

When substituting for the number of years it is important to use parentheses. That way we can just type what we see into the calculator:
5400 multiplication .03 multiplication (2 divide 3) =

Future Value

Rather than asking how much interest will be paid, we may want to know how much money will be paid at the end of the borrowing time (or investment period). This end amount is called the future value. To calculate the future value, we could just add the principal plus interest. There is also a future value formula for simple interest:

$\mathrm{FV} = \mathrm{P}\ (1 + \mathrm{rt})$

FV = future value for simple interest
P = principal or present value which is the amount borrowed
r = annual simple interest rate
t = time measured in years

We can talk about the accumulated amount, total amount paid, and the amount at the end of the investment. All are examples of future value. The best way to think of present value or principal and future value is with a time line.

Present Value ⟶ time invested or borrowed ⟶ Future Value
Earlier time Later in time

Examples

3. If a principal of \$2800 is invested for a period of 5 years at a simple interest rate of 3.4%, what will be the future value?

Steps	Reasons
$\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$	Recognize the type of interest and write future value for simple interest formula.
$\mathrm{FV} =$???	We are looking for the future value.
$\mathrm{P} =$ \$2800	We have the principal, P.
$\mathrm{r} =$ 3.4% or .034	The simple interest rate is given.
$\mathrm{t} =$ 5 years	The number of years that the money is borrowed.

$\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$ Substitute the numbers into the formula.
$\mathrm{FV} = 2800(1 + .034 \cdot 5)$
$\mathrm{FV} = \$3276$ Use a calculator to find the future value FV .

Be sure to write down every symbol in the formula, replace the variables without losing any of the parentheses or operations, and then type every symbol into the calculator.

4. If \$12,000 is borrowed for 120 days at 5.2% simple interest, how much must be paid back at the end?

Steps	Reasons
$\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$	We are asked for the end amount to be paid back at the end of the borrowing time. So, use the future value formula for simple interest.
$\mathrm{FV} =$???	We are looking for the future value.
$\mathrm{P} =$ \$12,000	The amount borrowed, P, comes before the time borrowed.
$\mathrm{r} =$ 5.2% or .052	The simple interest rate is given.
$\mathrm{t} =$ 120 days	The time that the money is borrowed.

Again we need to make an adjustment for the time. The 120 days needs to be converted to years. Often students will just divide the 120 days by 365 to get the number of years, which is correct. We can also use dimensional analysis.

$$\mathrm{t} = 120\ \mathrm{days} \cdot \frac{1\ \mathrm{year}}{365\ \mathrm{days}} = \frac{120}{365}\ \mathrm{years}$$

The fraction $\frac{1\ \mathrm{year}}{365\ \mathrm{days}}$ is the appropriate unit fraction because it has a value of 1. Sometimes 360 days is used in the denominator, which is a very close approximation.

$\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$ Substitute the numbers into the formula.

$\mathrm{FV} = 12{,}000\left(1 + .052 \cdot \left(\frac{120}{365}\right)\right)$ Use a calculator to find the future value FV.

$\mathrm{FV} =$ \$12,205.15 and \$12,205 Round off to the nearest penny or dollar.

It may be that we have all the information except the present value or simple interest rate or time. In those cases, once we have plugged the other values into the formula we can solve for the present value or simple interest rate or time algebraically.

Examples

5. If the future value is \$632.40 and the present value is \$600 for a 9 month investment, what is the simple interest rate?

Steps	Reasons
$\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$	We are asked about a problem involving future value for simple interest.
FV = 632.40	The future value is given.
P = \$600	The present value is given.
r = ???	Find the simple interest rate.
t = 9 months	The time that the money is borrowed.

Use dimensional analysis to change months to years.

$$\mathrm{t} = 9 \text{ months} \cdot \frac{1 \text{ year}}{12 \text{ months}} = \frac{9}{12} \text{ or } \frac{3}{4} \text{ or } 0.75 \text{ years}$$

$\mathrm{FV} = \mathrm{P}\,(1 + \mathrm{rt})$ Substitute the numbers into the formula.

$632.40 = 600(1 + \mathrm{r} \cdot 0.75)$ Now solve the equation for r. Begin with the distributive property because of the + inside the parentheses.

$632.4 = 600 + 450\mathrm{r}$

$632.4 - 600 = 450r$

$32.4 = 450\mathrm{r}$

$\frac{32.4}{450} = \mathrm{r}$

$\mathrm{r} = .072$ or 7.2% Write the simple interest rate as a percent.

6. How much money needs to be invested at 4.8% simple interest for three years to meet a future value goal of $17,000? Round-up to the nearest dollar.

Steps	Reasons
$\text{FV} = \text{P}\ (1 + \text{rt})$	We are asked about a problem involving future value for simple interest.
FV = $17,000	The future value is given.
P = ???	We are asked for the present value.
$\text{r} = 4.8\%\ \text{or}\ .048$	The simple interest rate is given.
t = 3 years	The time that the money is borrowed.
$\text{FV} = \text{P}\ (1 + \text{rt})$	Substitute the numbers into the formula.
$17{,}000 = \text{P}(1 + .048 \cdot 3)$ $17{,}000 = \text{P}(1.144)$ $\frac{17{,}000}{1.144} = \text{P}$ $\text{P} = 14{,}860.139\ldots$	Now solve the equation for r. Begin with the distributive property because of the + inside the parentheses.
$\text{P} = \$14{,}861$	

We are in the habit of always rounding-up when finding the present value. We do this upwards rounding for present value because that will assure that the future value goal is met. Generally, it is not of huge importance, but financial advisors should want to make sure that the goal is met or surpassed. If we round-down even by a little bit, then the goal is not quite met.

As we go through the different types of formulas, we will need to determine which formula we will need to use. Here the formulas talk about simple interest. We need to determine whether we want the interest, the future value (end amount or accumulated value), or the present value (the amount before the time the money grows).

Exercises

Solve and round to the nearest penny:

1. Find the simple interest for a principal of $10,400 that is borrowed at simple interest rate 4% for a period of 5 years.
2. Find the simple interest for a principal of $3000 that is borrowed at simple interest rate 2.5% for a period of 3 years.
3. Find the simple interest for a principal of $120,000 that is borrowed at simple interest rate 12.5% for a period of 15 years.
4. Find the simple interest for a principal of $15,000 that is borrowed at simple interest rate 9.25% for a period of 10 years.
5. Find the simple interest for a principal of $5,400 that is borrowed at simple interest rate 4% for a period of 9 months.
6. Find the simple interest for a principal of $17,200 that is borrowed at simple interest rate 3.5% for a period of 8 months
7. Find the simple interest for a principal of $20,500 that is borrowed at simple interest rate 4.25% for a period of 8 months.
8. Find the simple interest for a principal of $41,000 that is borrowed at simple interest rate 12% for a period of 3 months.
9. Find the simple interest for a principal of $15,000 that is borrowed at simple interest rate 4.5% for a period of 100 days.
10. Find the simple interest for a principal of $8,700 that is borrowed at simple interest rate 2.8% for a period of 60 days.
11. Find the simple interest for a principal of $325,000 that is borrowed at simple interest rate 12.5% for a period of 150 days.
12. Find the simple interest for a principal of $200,000 that is borrowed at simple interest rate 7.8% for a period of 45 days.

13. If a principal of $20,000 is invested for a period of 8 years at a simple interest rate of 3.7%, what will be the future value?

14. If a principal of $14,000 is invested for a period of 10 years at a simple interest rate of 5.2%, what will be the future value?

15. If a principal of $15,000 is invested for a period of 8 months at a simple interest rate of 7.5%, what will be the future value?

16. If a principal of $200,000 is invested for a period of 4 months at a simple interest rate of 6.5%, what will be the future value?

17. If a principal of $14,600 is invested for a period of 90 days at a simple interest rate of 10.5%, what will be the future value?

18. If a principal of $28,000 is invested for a period of 30 days at a simple interest rate of 7.25%, what will be the future value?

Find the simple interest rate:

19. If the future value is $9560 and the present value is $8000 for a 3 year investment, what is the simple interest rate?

20. If the future value is $15,125 and the present value is $12,500 for a 5 year investment, what is the simple interest rate?

21. If the future value is $18,342 and the present value is $18,000 for a 3 month investment, what is the simple interest rate?

22. If the future value is $24,252 and the present value is $23,500 for a 6 month investment, what is the simple interest rate?

Find the principal (also called present value). Round-up to the nearest dollar

23. How much money needs to be invested at 3.4% simple interest for six years to meet a future value goal of $15,400? Round-up to the nearest dollar.

24. How much money needs to be invested at 8.25% simple interest for eight years to meet a future value goal of $120,000? Round-up to the nearest dollar.

25. How much money is borrowed at 12.4% simple interest for 90 days if the amount paid after the 90 days is $9,000? Round-up to the nearest dollar.

26. How much money is borrowed at 3.25% simple interest for 120 days if the amount paid after the 120 days is $15,000? Round-up to the nearest dollar.

Answer the following, and round to the nearest dollar. If asked to find the principal (or present value), round-up to the nearest dollar.

27. Find the interest paid on a $5000 simple interest loan for 45 days at 11.25%.

28. How much money was invested eight years ago at 4.25% simple interest if today's value is $17,500?

29. What is the amount due on a $25,000 loan for 120 days at simple interest rate 9.9%?

30. Find the interest paid on an 80 day $125,000 loan at 8.5% simple interest.

31. What is the simple interest rate if a $7000 investment for five years has a future value of $8300? (Round to the nearest tenth of a percent.)

32. What is the simple interest rate if a $2850 investment for two years has a future value of $3150? (Round to the nearest tenth of a percent.)

33. How much money was invested twenty years ago at 5.25% simple interest if today's value is $100,000?

34. What is the amount due on a $5,000 loan for 90 days at simple interest rate 12.2%?

With simple interest the interest is applied only once at the end of the time invested or borrowed. The more typical situation is compound interest where the interest is applied after a certain amount of time. Then that interest grows for the rest of time. For instance, most regular savings accounts have interest compounded daily. After depositing money into the account at the end of the first day, a little bit of interest is earned. That little bit of interest is then in the account and it also earns interest for the rest of the time that the money is in the account. At the end of the second day a little more interest is earned, which also grows for the rest of the time that the money is in the account. The process continues where the interest earned is put into the account daily to grow for the rest of the time.

We have a formula for the future value for compound interest:

$$\mathrm{FV} = \mathrm{P}\left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$$

$\mathrm{FV} =$ future value for compound interest
$\mathrm{P} =$ principal or present value
$\mathrm{r} =$ annual compound interest rate
$\mathrm{t} =$ time measured in years
$\mathrm{n} =$ number of compounding per year

From simple interest we know what most of the letters stand for, but we should take a closer look at the value of n.

Compounded daily	n = 365 or 360 to round-off
Compounded monthly	n = 12
Compounded quarterly	n = 4
Compounded semi-annually	n = 2
Compounded annually	n = 1

Examples:

1. $15,000 is invested for six years at 4.7% interest compounded monthly. What is the future value?

Steps	Reasons
$\mathrm{FV} = \mathrm{P}\left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$	Recognize the type of interest. We see the words compound interest and we are asked for future value.
$\mathrm{FV} =$???	We are looking for the future value.
$\mathrm{P} =$ \$15,000	We have the principal, P.
$\mathrm{r} =$ 4.7% or .047	The compound interest rate is given.
$\mathrm{t} =$ 6 years	The number of years that the money is borrowed.
n = 12	Compounded monthly means n = 12 for 12 times per year.
$\mathrm{FV} = \mathrm{P}\left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$	Substitute the numbers into the formula. Use a calculator to find the future value FV .

$$\mathrm{FV} = 15{,}000\left(1+\frac{.047}{12}\right)^{12\cdot 6}$$

The future value is $19,875.73. Round the future value to the nearest penny or dollar.

2. What is the accumulated value for a $50,000 loan for 3.5 years at 6% interest compounded daily?

Steps	Reasons
$\mathrm{FV} = \mathrm{P}\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$	Recognize the type of interest. We see the words compound interest and we are asked for the accumulated value, which is the future value.
$\mathrm{FV} = ???$	We are looking for the future value.
$\mathrm{P} = \$50{,}000$	We have the principal, P.
$\mathrm{r} = 6\%$ or $.06$	The compound interest rate is given.
$\mathrm{t} = 3.5$ years	The number of years that the money is borrowed.
$\mathrm{n} = 365$	Compounded daily means $\mathrm{n} = 365$ for 365 times per year. Sometimes $\mathrm{n} = 360$ is used.
$\mathrm{FV} = \mathrm{P}\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$	Substitute the numbers into the formula.
$\mathrm{FV} = 50{,}000\left(1+\frac{.06}{365}\right)^{365\cdot 3.5}$	Use a calculator to find the future value FV .

The accumulated value is $61,682.84.

Less frequently we may see the future value for interest <u>compounded continuously</u>. Imagine that instead of daily we had the compounding period every hour or every minute or every second or every part of a second. Then we would be approaching continuous compounding. The formula for continuous compound interest is

$\mathrm{FV} = \mathrm{Pe}^{\mathrm{rt}}$

$\mathrm{FV} =$ future value for continuous compound interest
$\mathrm{P} =$ principal or present value
$\mathrm{r} =$ annual continuous compound interest rate
$\mathrm{t} =$ time measured in years

The letter e is actually a number. It is an irrational number like the number π. Since e written as a decimal number goes on forever without repeating, we have to use a symbol to express the number exactly. The number e is approximately equal to 2.71828…

Example:

3. Inflation is often calculated using continuous compound interest. If the inflation rate is 2.5% compounded continuously, how much will a $200 cart of groceries cost in 20 years?

Steps	Reasons
$FV = Pe^{rt}$	Recognize the type of interest. We see the words continuous compound interest and we are asked for cost in 20 years, which is the future value.
$FV =$???	We are looking for the future value.
$P =$ $200	We have the principal or present value, P.
$r =$ 2.5% or .025	The continuous compound interest rate is given.
$t =$ 20 years	The number of years.
$FV = Pe^{rt}$ $FV = 200e^{.025 \cdot 20}$	Substitute the numbers into the formula. Use a scientific calculator to find the future value A.

The cart of groceries will cost $329.74 in twenty years.

As with simple interest, we may have the future value and want to know the present value for compound interest. Rather than using the future value formula and solving for the present value like we did with simple interest, we will use a present value formula for compound interest, which is just an algebraic manipulation of the future value formula for compound interest that we have above.

Present value for compound interest:

$$P = \frac{FV}{\left(1 + \frac{r}{n}\right)^{nt}}$$

$FV =$ future value for compound interest
$P =$ principal or present value
$r =$ annual compound interest rate
$t =$ time measured in years
$n =$ number of compounding per year

Examples:

4. A salesperson receives a large bonus for having the best sales record for the year. If she wants to have $25,000 in five years for the down payment on a house, how much must she put aside now at 7.5% compounded quarterly?

Steps	Reasons
$P=\frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$	Recognize the type of interest. We see the words compound interest and we are asked for the amount now before the investment. We are looking for the present value with compound interest.
$FV = \$25,000$	We have the future value FV.
$P = ???$	We are looking for the present value, P.
$r = 7.5\%$ or $.075$	The compound interest rate is given.
$t = 5$ years	The number of years that the money is borrowed.
$n = 4$	Compounded quarterly means $n = 4$ for 4 times per year.
$P=\frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$	Substitute the numbers into the formula. Use a calculator to find the present value P .
$P=\frac{25,000}{\left(1+\frac{.075}{4}\right)^{4\cdot 5}}$	

The salesperson needs to put aside \$17,242.

For present value problems we should always round-up to the nearest dollar or nearest penny. That way we meet or exceed the goal. If rounding leads to rounding-down, then we just miss the stated goal.

5. A twenty-five year old believes he will need \$750,000 to retire comfortably. How much will he need to put aside now at 3.25% interest compounded monthly to meet his goal in 40 years?

Steps	Reasons
$P=\frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$	Recognize the type of interest. We see the words compound interest and we are asked for the amount now before the investment. We are looking for the present value with compound interest.
$FV = \$750,000$	We have the future value goal, FV.
$P = ???$	We are looking for the present value, P.
$r = 3.25\%$ or $.0325$	The compound interest rate is given.
$t = 40$ years	The number of years that the money is borrowed.
$n = 12$	Compounded quarterly means $n = 12$ for 12 times per year.
$P=\frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$	Substitute the numbers into the formula. Use a calculator to find the present value, P .

$$\mathrm{P} = \frac{750{,}000}{\left(1 + \frac{.0325}{12}\right)^{12\cdot 40}}$$

The person needs to put aside $204,758.34 in order to meet the retirement goal.

That is amazing! By putting aside $204,758.34 at 3.25% interest compounded monthly, a twenty-five year old can meet a retirement goal of $750,000. The other $545,241.66 all comes from interest earned on the investment. The interest is the future value minus the present value (750,000-204,758.34 = 545,241.66) because the present value is the amount put in to the account. So, why doesn't everybody just put aside about $200,000 when they are 25 years old? What do we do instead? In the next chapter we will learn about savings plans with periodic (monthly) payments called annuities.

Exercises

For future value problems, round to the nearest dollar. For present value problems, round-up to the nearest dollar.

1. $18,000 is invested for ten years at 3.7% interest compounded monthly. What is the future value?

2. $75,000 is invested for four years at 1.5% interest compounded monthly. What is the future value?

3. What is the accumulated value for a $150,000 loan for twenty-five years at 5.25% interest compounded daily?

4. What is the accumulated value for an $80,000 loan for eight years at 6% interest compounded daily?

5. $150,000 is invested for seventeen years at 4.75% interest compounded semiannually. What is the future value?

6. $5,000 is invested for 5.5 years at 4.1% interest compounded semiannually. What is the future value?

7. What is the accumulated value for a $10,000 loan for twelve years at 8% interest compounded quarterly?

8. What is the accumulated value for a $500,000 loan for fifteen years at 5.9% interest compounded quarterly?

9. $30,000 is invested for twenty years at 11.5% interest compounded monthly. What is the future value?

10. $320,000 is invested for fifteen years at 1.5% interest compounded monthly. What is the future value?

11. $60,000 is invested for twenty years at 6.4% interest compounded annually. What is the future value? How is annual compounding different from simple interest?

12. What is the accumulated value for a $45,000 loan for fifteen years at 7.25% interest compounded annually? How is annual compounding different from simple interest?

13. What is the accumulated value for a $95,000 loan for 8.5 years at 3.25% interest compounded daily?

14. What is the accumulated value for a $190,000 loan for five years at 2% interest compounded daily?

15. $50,000 is invested for fourteen years at 2.75% interest compounded semiannually. What is the future value?

16. $40,000 is invested for 4.5 years at 1.75% interest compounded semiannually. What is the future value?

17. What is the accumulated value for an $800,000 loan for four years at 7.5% interest compounded quarterly?

18. What is the accumulated value for a $1,500,000 loan for twenty years at 6.8% interest compounded quarterly?

19. $32,000 is invested for ten years at 4.5% interest compounded continuously. What is the future value?

20. $102,000 is invested for twenty-two years at 3.5% interest compounded continuously. What is the future value?

21. What is the accumulated value for a $7,000 loan for four years at 5.5% interest compounded continuously?

22. What is the accumulated value for a $68,000 loan for twelve years at 3.75% interest compounded continuously?

23. Inflation is often calculated using continuous compound interest. The cost of a mixture of items and the salary that it takes to purchase these items grows continuously. If the inflation rate is 2.5% compounded continuously, how large of a salary will somebody need in thirty years to have the same buying power as a $30,000 salary in today’s dollars?

24. Inflation is often calculated using continuous compound interest. The cost of a mixture of items and the salary that it takes to purchase these items grows continuously. If the inflation rate is 1.5% compounded continuously, how large of a salary will somebody need in forty years to have the same buying power as a $25,000 salary in today's dollars?

25. If the inflation rate is 2.25% compounded continuously and the price of gasoline follows this inflation rate, how much will a $50 tank of gasoline cost in ten years?

26. If the inflation rate is 1.85% compounded continuously and the price of groceries follows this inflation rate, how much will a $175 cart of groceries cost in twenty years?

27. An investment grows at 5% compounded daily for ten years. If the future value of the investment is $40,000, what is the present value?

28. An investment grows at 3.25% compounded daily for fifteen years. If the future value of the investment is $25,000, what is the present value?

29. An investment grows at 4.2% compounded quarterly for twelve years. If the future value of the investment is $100,000, what is the present value?

30. An investment grows at 7.5% compounded quarterly for eight years. If the future value of the investment is $100,000, what is the present value?

31. How much money must be invested today at 6.7% interest compounded monthly so that there is an accumulated value of $50,000 in five years?

32. How much money must be invested today at 3.4% interest compounded monthly so that there is an accumulated value of $40,000 in fifteen years?

33. How much money can be borrowed at 3.65% interest compounded semiannually, if the borrower is willing to pay back $75,000 in ten years?

34. How much money can be borrowed at 4.25% interest compounded semiannually, if the borrower is willing to pay back $15,000 in two years?

35. A salesperson receives a large bonus for having the best sales record for the year. If she wants to have $15,000 in three years for the down payment on a house, how much must she put aside now at 8.5% compounded quarterly?

36. A salesperson receives a large bonus because the company meets all of the objectives and she has skills that the company feels are irreplaceable. If she wants to have $40,000 in six years for the down payment on a house, how much must she put aside now at 9.5% compounded quarterly?

37. If today's value of a government savings bond is $20,000, how much was invested seven years ago? The interest rate was 6.5% compounded daily.

38. If today's value of a government savings bond is $3,000, how much was invested ten years ago? The interest rate was 3.5% compounded daily.

39. A forty year old believes he will need $1,000,000 to retire comfortably. How much will he need to put aside now at 4.25% interest compounded semiannually to meet his goal in 25 years?

40. A twenty year old believes he will need $2,000,000 to retire comfortably. How much will he need to put aside now at 3.25% interest compounded semiannually to meet his goal in 45 years?

41. A young person is trying to figure out how to retire comfortably, which she believes will take $1,500,000.

    a. How much must a twenty-five year old put aside for 40 years at 5% interest compounded monthly to reach the goal?

    b. How much must a forty-five year old put aside for 20 years at 5% interest compounded monthly to reach the goal?

    c. How much must the twenty-five year old put aside for 40 years at 7% compounded monthly to reach the goal? 3% compounded monthly?

    d. The twenty-five year old may have other financial obligations – family, housing, school loans to pay back. What else can be done aside from putting aside a huge amount of money all at once?

42. A young person is trying to figure out how to retire comfortably, which he believes will take $2,000,000.

    a. How much must a twenty year old put aside for 45 years at 4.3% interest compounded quarterly to reach the goal?

    b. How much must a forty-five year old put aside for 20 years at 4.3% interest compounded quarterly to reach the goal?

    c. How much must the twenty year old put aside for 45 years at 7.5% compounded quarterly to reach the goal? 2.8% compounded quarterly?

d. The twenty year old may have little money and other financial obligations – food, housing, credit card debt, living the good life. What else can be done aside from putting aside a huge amount of money all at once?

43. A teacher decides to put aside some money to have for a rainy day, which fortunately never seems to come. First he puts aside $10,000 for five years at 2.25% compounded monthly interest. After five years the economy changes and he is able to take that money and put it into an account that earns 6.4% interest compounded semiannually for twenty years. How much does the teacher have at the end of the twenty-five year investment time?

44. A doctor decides to put aside some money to have for a rainy day, which fortunately never seems to come. First she puts aside $25,000 for three years at 3.25% interest compounded monthly interest. After three years the economy changes and she is able to take that money and put it into an account that earns 6.4% interest compounded quarterly for fifteen years. How much does the doctor have at the end of the eighteen year investment time?

Often we do not have enough money to meet a future financial goal all at one time. An annuity is a fixed sum paid over equal periods per year over some number of years at a set interest rate. A very relevant example for most people is their retirement account. For instance, somebody saving for retirement may put aside \$400 per month for thirty years. If the interest and payment schedule stay the same, we are talking about an annuity.

Formula for future value of an annuity:

$$\mathrm{FV} = \frac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$$

FV = future value for an annuity
Pmt = periodic payment
r = annual compound interest rate
t = time measured in years
n= number of payments, which is the same as the number of compounding per year

Examples:

1. If \$400 is put aside every month at 5% interest compounded monthly for 30 years, what is the accumulated value? How much of that accumulated value is interest?

Steps	Reasons
$\mathrm{FV} = \frac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$	Since we are putting aside a set amount each month for thirty years at a constant interest rate, we have an annuity.
FV = ???	We are looking for the accumulated value or future value of the annuity.
Pmt = \$400	The periodic payment is \$400 per month
r = 5% or .05	5% annual compound interest rate
t = 30 years	the time was 30 years
n= 12	There are 12 monthly payments, which is the same as the number of compounding per year
$\mathrm{FV} = \frac{400\left[\left(1+\frac{.05}{12}\right)^{12\cdot 30}-1\right]}{\left(\frac{.05}{12}\right)}$	Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.
$\mathrm{FV} = \$332{,}903.45$	Carefully, push all the buttons on the calculator.

Look below for the steps that will work with many scientific calculators.

The next question is how much of the money came from interest. To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

Steps	Reasons
Deposits = 400·12·30 Deposits = $144,000	To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 30 years.
Interest = future value – deposits Interest = 332,903.45 – 144,000	The interest is the amount of money at the end minus the amount of money deposited.

The interest is $188,903.45

Many scientific calculators require us to push the following buttons:

Formula

$$\mathrm{FV} = \frac{400\left[\left(1+\frac{.05}{12}\right)^{12\cdot 30}-1\right]}{\left(\frac{.05}{12}\right)}$$

Buttons

400
(open parentheses twice like the formula
(
1
Plus
.05
Divide
12
) close parentheses
^ or x^{y} use the exponent button
360 or $(12 \cdot 30)$ type either way
Minus
1
)
Divide
(
.05
Divide
12
)
=

To get out of the exponent area some calculators require pushing a right arrow.

2. A young person is deciding whether to start investing now or wait. After all, he is young, has many expenses like cars and a house, and wants to enjoy his money while being young.

a. If interest is 6% compounded monthly, how much money will the investor have if he invests $500 per month for 40 years? How much of that money comes from interest?

b. If the interest is kept at 6% compounded monthly, how much money will the investor have if he puts aside $1000 per month for 20 years? How much of that money will be interest?

a. Consider the first scenario.

Steps	Reasons
$\mathrm{FV} = \dfrac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$	Since the investor is putting aside a set amount each month for forty years at a constant interest rate, we have an annuity.
$\mathrm{FV} =$???	We are looking for the accumulated value or future value of the annuity.
$\mathrm{Pmt} = \$500$	The periodic payment is $500 per month
$\mathrm{r} = 6\%$ or .06	Assuming 6% annual compound interest rate
$\mathrm{t} =$ 40 years	the time was 40 years
$\mathrm{n}=$ 12	There are 12 monthly payments, which is the same as the number of compounding per year
$\mathrm{FV} = \dfrac{500\left[\left(1+\frac{.06}{12}\right)^{12\cdot 40}-1\right]}{\left(\frac{.06}{12}\right)}$	Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.
$\mathrm{FV} = \$995{,}745.37$	Carefully, push all the buttons on the calculator as outlined above.

Now find the interest.

Steps	Reasons
Deposits = 500·12·40 Deposits = $240,000	To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 40 years.
Interest = future value – deposits Interest = 995,745.37 – 240,000	The interest is the amount of money at the end minus the amount of money deposited.

The interest is $755,745.37

b. Consider the second scenario.

Steps	Reasons
$\mathrm{FV} = \dfrac{\mathrm{Pmt}\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}{\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}$	Since he is putting aside a set amount each month for thirty years at a constant interest rate, we have an annuity.
$\mathrm{FV} =$???	We are looking for the accumulated value or future value of the annuity.
$\mathrm{Pmt} = \$1000$	The periodic payment is $1000 per month
$\mathrm{r} = 6\%$ or .06	6% annual compound interest rate
$\mathrm{t} =$ 20 years	the time is 20 years
$\mathrm{n} =$ 12	There are 12 monthly payments, which is the same as the number of compounding per year
$\mathrm{FV} = \dfrac{1000\left[\left(1+\frac{.06}{12}\right)^{12\cdot 20}-1\right]}{\left(\frac{.06}{12}\right)}$	Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.
$\mathrm{FV} = \$462{,}040.90$	Carefully, push all the buttons on the calculator as in example 1.

To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

Steps	Reasons
Deposits = 1000·12·20 Deposits = $240,000	To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 20 years.
Interest = future value – deposits Interest = 462,040.90 – 240,000	The interest is the amount of money at the end minus the amount of money deposited.

The interest is $222,040.90

In both the 20 year and the 40 year examples, the investor puts aside the same total amount of money ($240,000). By starting sooner as in scenario a, the investor gains an extra $533,704.47 all in interest.

Periodic Deposits of an Annuity

We may want to how much money we need to set put aside regularly to reach a future financial goal. If we put the same restrictions that we had before of periodic payments with a constant interest rate, then we get a formula for the periodic payment based on a known goal. Most of us will be interested in our retirement and perhaps our children's education. If we know the future amount that we want, then we can calculate the amount that needs to be put aside regularly.

Formula for periodic payment of an annuity:

$$\mathrm{Pmt} = \frac{\mathrm{FV}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}$$

Pmt = periodic payment
FV = future value for an annuity
r = annual compound interest rate
t = time measured in years
n= number of payments, which is the same as the number of compounding per year

Examples:

3. When studying compound interest, we saw that a twenty-five year old would have to put aside a bit over two hundred thousand dollars all at once for 40 years at 3.25% monthly interest to reach a retirement goal of $750,000 to retire comfortably. While that is an amazing amount of interest unfortunately many of us do not have that kind of money on hand. How much does the twenty-five year old need to put aside each month to have $750,000 in 40 years at 3.25% compounded monthly? How much of that amount is interest?

Steps	Reasons
$\mathrm{Pmt} = \dfrac{\mathrm{FV}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}-1\right]}$	Since we are putting aside a set amount each month for forty years at a constant interest rate, we have an annuity.
$\mathrm{Pmt} = ???$	We are looking for the periodic (monthly) payment
FV = 750,000	The future value of the annuity is the goal of $750,00
r = 3.25% or .0325	3.25% annual compound interest rate
t = 40 years	The time was 40 years
n= 12	There are 12 monthly payments, which is the same as the number of compounding per year
$\mathrm{Pmt} = \dfrac{750{,}000\left(\frac{.0325}{12}\right)}{\left[\left(1+\frac{.0325}{12}\right)^{12\cdot 40}-1\right]}$	Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.
$\mathrm{Pmt} = \$762.81$	Carefully, push all the buttons on the calculator.

The next question is how much of the money came from interest. To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

Steps	Reasons
Deposits = 762.81·12·40 Deposits = $366,148.80	To find the amount of money that is put into the annuity, we take the monthly payment multiply by 12 months per year and then multiply by 40 years.
Interest = future value – deposits Interest = 750,000 – 366,148.80	The interest is the amount of money at the end minus the amount of money deposited.

The interest is $383,851.20

4. A company wants to make an extra retirement fund for its executives so that the CEO and other top officers have an extra $10,000,000 to share. How much money does the company need to put aside quarterly for 15 years at 5.25% interest compounded quarterly to meet this goal? How much of that goal is deposits and how much is interest?

Steps	Reasons
$\text{Pmt} = \dfrac{\text{FV}\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}$	Since we are putting aside a set amount each month for 15 years at a constant interest rate, we have an annuity.
Pmt = ???	We are looking for the periodic (quarterly) payment
FV = 10,000,000	The future value of the annuity is the goal of $10,000,00
r = 5.25% or .0525	5.25% annual compound interest rate
t = 15 years	The time is 15 years
n= 4	There are 4 quarterly payments, which is the same as the number of compounding per year
$\text{Pmt} = \dfrac{10{,}000{,}000\left(\frac{.0525}{4}\right)}{\left[\left(1+\frac{.0525}{4}\right)^{4\cdot 15}-1\right]}$	Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.
$\text{Pmt} = \$110{,}604.24$	Carefully, push all the buttons on the calculator.

The next question is how much of the money came from interest. To get the interest, find out how much of the money came from deposits and subtract it from the end value of the annuity.

Steps	Reasons
Deposits = 110,604.24·4·15 Deposits = $6,636,254.40	To find the amount of money that is put into the annuity, we take the monthly payment multiply by 4 months per year and then multiply by 15 years.
Interest = future value – deposits Interest = 10,000,000 – 6,636,254,4	The interest is the amount of money at the end minus the amount of money deposited.

The interest is $3,363,745.60

There are several ways to increase the future value of an annuity:
Higher interest
Larger deposits
Longer time investing
We cannot control the interest rates that we receive, but we can start saving for our retirement while we are young rather than waiting.

Exercises

For the following, round to the nearest dollar.

1. If \$250 is put aside every month at 3.5% interest compounded monthly for 30 years, what is the accumulated value? How much of that accumulated value is interest?

2. If \$450 is put aside every month at 7.5% interest compounded monthly for 25 years, what is the accumulated value? How much of that accumulated value is interest?

3. If \$600 is put aside every month at 6.8% interest compounded monthly for 40 years, what is the accumulated value? How much of that accumulated value is interest?

4. If \$520 is put aside every month at 3.2% interest compounded monthly for 35 years, what is the accumulated value? How much of that accumulated value is interest?

5. A young person is deciding whether to start investing now or wait. After all, she is young, has many expenses like cars and a house, and wants to enjoy her money while being young.
   a. If interest is 4.5% compounded monthly, how much money will the investor have if she invests \$450 per month for 40 years? How much of that money comes from interest?
   b. If the interest is kept at 4.5% compounded monthly, how much money will the investor have if she puts aside \$900 per month for 20 years? How much of that money will be interest?
   c. In both cases the same amount of money is put aside. Does the extra interest make it worth starting to save for retirement early?

6. A young person is deciding whether to start investing now or wait. After all, he is young, has many expenses like cars and a house, and wants to enjoy his money while being young.
   a. If interest is 5.8% compounded monthly, how much money will the investor have if he invests \$600 per month for 40 years? How much of that money comes from interest?
   b. If the interest is kept at 5.8% compounded monthly, how much money will the investor have if he puts aside \$1200 per month for 20 years? How much of that money will be interest?

   c. In both cases the same amount of money is put aside. Does the extra interest make it worth starting to save for retirement early?

7. How much does a thirty year old need to put aside each month to have \$1,000,000 in 35 years at 6.25% interest compounded monthly? How much of that amount is interest?

8. How much does a twenty year old need to put aside each month to have \$1,500,000 in 45 years at 7.25% interest compounded monthly? How much of that amount is interest?

9. How much does a fifty year old need to put aside each month to have \$1,000,000 in 15 years at 4.5% interest compounded monthly? How much of that amount is interest?

10. How much does an eighteen year old need to put aside each month to have \$2,000,000 in 47 years at 5.25% interest compounded monthly? How much of that amount is interest?

11. A couple wants to save for their child’s education. Since education costs are always rising, they decide to try to save \$200,000. If the child does not want to go to college, the couple figures that they can get a lake house.
    a. How much money does the couple need to put aside each month for 18 years at 5.25% interest compound monthly to reach their goal?
    b. How much money does the couple need to put aside each month for 9 years at 5.25% interest compound monthly to reach their goal?
    c. Is it important for the couple to start early to reach their goal and pay for their child’s education?
    d. How much money does the couple need to put aside each month for 18 years at 9.25% interest compound monthly to reach their goal? At 2.25% compounded monthly?

12. A couple wants to save for their child’s education. Since education costs are always rising, they decide to try to save \$150,000. If the child does not want to go to college, the couple figures that they can use the money to help with their retirement.
    a. How much money does the couple need to put aside each month for 18 years at 4.25% interest compound monthly to reach their goal?
    b. How much money does the couple need to put aside each month for 9 years at 4.25% interest compound monthly to reach their goal?

c. Is it important for the couple to start early to reach their goal and pay for their child’s education?
d. How much money does the couple need to put aside each month for 18 years at 8.25% interest compound monthly to reach their goal? At 3.25% compounded monthly?

13. A young bachelor feels that he spends the majority of his paycheck on tobacco, alcohol, and hitting the town with his friends. He figures that if he quits smoking, only goes out on the weekend, and limits dining out to special occasions, he will be able to save $350 per month. If he puts that money into an annuity that earns 4.75% interest compounded monthly, how much will he have saved after 35 years? How much of that money is interest?

14. A chronic gambler seeks financial help and discovers that she is losing $200 per month by playing the slot machines. If she saves $200 per month at 6.25% interest compounded monthly, how much will she have saved after 25 years? How much of that money comes from interest?

15. A company wants to make an extra retirement fund for its employees so that the employees who work for the company more than twenty years will get a large bonus upon retiring. If the company wants to save $20,000,000 in 10 years, how much money does the company need to put aside quarterly at 4.25% interest compounded quarterly to meet this goal? How much of the goal comes from the deposits and how much is interest?

16. A company wants to save $200,000,000 for future acquisitions. How much money must the company put aside quarterly for 15 years at 6.5% interest compounded quarterly? How much of the goal comes from the deposits and how much is interest?

For most of us our largest purchase will be a house. We may end up buying more than one house, but it is unlikely that we will buy more than two or three houses. The bank's loan officer will be the expert. He will be trained and may have processed hundred's of loans by the time you talk to him about your mortgage. Making sure that you are informed of the options and their consequences will be the first step in making sure that you get the best possible deal with the bank when you buy your house.

A mortgage is an amount of money borrowed over a certain amount of time, which is paid back with periodic payments. There are two types of mortgages: variable and fixed interest mortgages. Variable interest may offer a lower rate in the beginning, but that rate can change over the time of the mortgage depending on current economic conditions. If the mortgage rate goes up too much, the borrower may find himself in a situation where he cannot afford the payments. A fixed interest mortgage has a set interest rate for the life of the loan. The interest rate may be slightly higher in the beginning, but the borrower does not take the chance the interest rates increase along with the periodic payments.

Formula for <u>periodic mortgage payments with fixed interest</u>:

$$\mathrm{Pmt} = \frac{\mathrm{B}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[1 - \left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{-\mathrm{nt}}\right]}$$

$\mathrm{Pmt} =$ periodic payment
$\mathrm{B} =$ the amount of money borrowed
$\mathrm{r} =$ annual compound interest rate
$\mathrm{t} =$ time measured in years
$\mathrm{n} =$ number of payments, which is the same as the number of compounding per year

The most important examples for us will involve the purchase of a house with monthly payments within our possibilities. We should want to know under what conditions we can save the most money. Some other important definitions for mortgages are down payment and points.

The <u>down payment</u> is a percent of the selling price, which is paid at the time the house is bought. Subtracting the down payment from the selling price will give us the amount borrowed, which is B in our above formula.

<u>Points</u> are a fee paid to the bank. Each point is equal to 1% of the amount borrowed. So, 2.5 points is 2.5% of the amount being borrowed. Points may be required by the bank. Sometimes points can be paid to get more favorable conditions such as a lower interest rate.

<u>Examples:</u>

1. A couple wanting to buy a house in the Midwest has finally found a house that they think is right for them. The house costs \$300,000 and can be financed with a fixed rate mortgage of 30 years at 4.8% compounded monthly. They decide to make a 5% down payment, and 1.5 points must be paid to the bank at closing.

a. What is the down payment?

Steps	Reasons
5% of 300,000	Down payment is a percent of the selling price
.05(300,000)	that is paid to the seller on the spot.

The down payment is \$15,000.

b. What is the amount borrowed?

Steps	Reasons
300,000 – 15,000	The couple needs to borrow the selling price minus
\$285,000 is borrowed.	the down payment, which is already paid to the seller.

c. What is the price of the 1.5 points at closing?

Steps	Reasons
1.5% of 285,000	1.5 points represents 1.5% of the amount borrowed.
.015(285,000)	These points should be thought of as a bank fee.

\$4275 paid for points.

d. What are the monthly payments?

Steps	Reasons
$$\mathrm{Pmt} = \frac{\mathrm{B}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[1-\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{-\mathrm{nt}}\right]}$$	We are paying back a loan with periodic (monthly) payments.

Pmt = ???	Pmt = we are looking for the monthly mortgage payment
B = \$285,000	The amount borrowed is \$285,000. 4.8% annual
r = 4.8% or .048	compound interest rate. The time is 30 years.
t = 30 years	There are 12 monthly payments, which is the same as
n= 12	the number of compounding per year.

$$\text{Pmt} = \frac{285{,}000\left(\frac{.048}{12}\right)}{\left[1-\left(1+\frac{.048}{12}\right)^{-12\cdot 30}\right]}$$

Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.

$\text{Pmt} = \$1495.30$ Carefully, push all the buttons on the calculator.

The monthly mortgage payment is $1495.30.

e. How much interest is paid?

To get the interest, find out how much of the money was paid altogether and subtract the amount borrowed.

Steps	Reasons
Total amount paid = 1495.30·12·30 Deposits = $538,308	To find the total amount of money that is paid to the bank, we take the monthly payment multiply by 12 months per year and then multiply by 30 years.
Interest = total amount paid – amount borrowed Interest = 538,308 – 285,000	The interest is the total amount of paid minus the amount borrowed.

The interest is $253,308.

A quarter of a million dollars just in interest is a lot to pay. Think of saving a quarter of a million dollars and then handing it over to the bank. What can the couple do? Let's take a look at the couple buying the same house, but under some circumstances that will save them money.

2. The couple wants to buy the same house in the Midwest, which they think is just right for them. The house costs $300,000, but this time they will finance it with a fixed rate mortgage of 10 years at 4.3% compounded monthly. They decide to make a 25% down payment, and 2 points must be paid to the bank at closing. How much must be paid for the points at closing? What is the monthly mortgage payment? What is the total interest?

To find the amount paid for points and the monthly mortgage payment we need the amount borrowed. So, first we calculate the down payment and subtract it from the price of the house.

Steps	Reasons
25% of 300,000 .25(300,000)	Down payment is a percent of the selling price that is paid to the seller on the spot.
The down payment is $75,000.	
300,000 – 75,000	The couple needs to borrow the selling price minus the down payment, which is already paid to the seller.
$225,000 is borrowed.	
2% of 225,000 .02(225,000)	2 points represents 2% of the amount borrowed. These points should be thought of as a bank fee.
$4500 paid for points.	
$\mathrm{Pmt} = \dfrac{\mathrm{B}\left(\frac{\mathrm{r}}{\mathrm{n}}\right)}{\left[1-\left(1+\frac{\mathrm{r}}{\mathrm{n}}\right)^{-\mathrm{nt}}\right]}$	We are paying back a loan with periodic (monthly) payments.
$\mathrm{Pmt} =$???	$\mathrm{Pmt} =$ we are looking for the monthly mortgage payment
$\mathrm{B} =$ $225,000	The amount borrowed is $225,000.
$\mathrm{r} =$ 4.3% or .043	4.3% annual compound interest rate
$\mathrm{t} =$ 10 years	The time is 10 years.
$\mathrm{n} =$ 12	There are 12 monthly payments, which is the same as the number of compounding per year.
$\mathrm{Pmt} = \dfrac{225{,}000\left(\frac{.043}{12}\right)}{\left[1-\left(1+\frac{.043}{12}\right)^{-12\cdot 10}\right]}$	Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.
$\mathrm{Pmt} =$ $2310.23	Carefully, push all the buttons on the calculator.
The monthly mortgage payment is $2310.23	
Total amount paid = 2310.23·12·10 Deposits = $277,227.60	To find the total amount of money that is paid to the bank, we take the monthly payment multiply by 12 months per year and then multiply by 10 years.
Interest = total amount paid–amount borrowed Interest = 277,227.60 – 225,000	The interest is the total amount of money paid minus the amount borrowed.
The interest is $52,227.60.	

In both examples the amount of money paid for points is about the same, but with the 10 year mortgage there is a huge savings in interest. With the 30 year mortgage in the first example the interest is $253,308. In the second example with a 10 year mortgage, the interest is $52,227.60. That means that the couple saves over $200,000 with the 10 year mortgage. By paying a larger down payment with a shorter borrowing time, you too can save more money than you may have thought. Remember to ask the loan officer what will be the monthly payment for a 10 year, 20 year, and 30 year loans. Calculate what the total interest will be for the life of the loan for various scenarios. Try to save up so that you have a larger down payment even if the bank does not require it.

Many other loans work like mortgages. Loans to buy a car, a boat, a kitchen appliance, or just about anything else may work like a mortgage. Although you will not pay points, you may pay a down payment with monthly payments.

Example:

3. A fisherman wants to buy a top of the line boat that costs $30,000. To help him pay for it, the boat company offers to let him pay a 10% down payment with monthly payments at 8% interest compounded monthly for 10 years. What are the monthly payments? How much interest is paid to finance the boat?

To find the monthly boat payment we need the amount borrowed. So, first we calculate the down payment and subtract it from the price of the boat.

Steps	Reasons
10% of 30,000 .10(30,000)	Down payment is a percent of the selling price that is paid to the seller on the spot.

The down payment is $3,000.

30,000 – 3,000 $27,000 is borrowed.	The fisherman will borrow the selling price of the boat minus the down payment, which is already paid to the seller.
$\text{Pmt} = \dfrac{B\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$	We need to find the monthly payments. So, we use the formula for periodic payments with fixed interest.
$\text{Pmt} =$???	$\text{Pmt} =$ we are looking for the monthly boat payment
$B =$ $27,000	The amount borrowed is $27,000.
$r =$ 8% or .08	8% annual compound interest rate
$t =$ 10 years	The time is 10 years.
$n =$ 12	There are 12 monthly payments per year, which is the same as the number of compounding per year.

$$\text{Pmt} = \frac{27{,}000\left(\frac{.08}{12}\right)}{\left[1-\left(1+\frac{.08}{12}\right)^{-12\cdot 10}\right]}$$

Replace the variables with the numbers and be sure to keep all parentheses and arithmetic symbols.

$\text{Pmt} =$ \$327.58

Carefully, push all the buttons on the calculator.

<u>The monthly boat payment is \$327.58</u>

Total amount paid =
327.58·12·10
Deposits = \$39,309.60

To find the total amount of money that is paid, we take the monthly payment multiply by 12 months per year and then multiply by 10 years.

Interest = total amount paid–amount borrowed
Interest = 39,309.60 – 27,000

The interest is the total amount of money paid minus the amount borrowed.

<u>The interest is \$12,309.60.</u>

Here the fisherman is paying an extra \$9,309.60 in interest to buy the boat. If we finance our purchases at high interest rates, we will always end up paying more for the items that what we buy.

Exercises

For the following questions, round to the nearest dollar.

1. A couple wants to buy a house that costs $275,000. The house can be financed with a fixed rate mortgage of 30 years at 5.25% compounded monthly. They decide to make a 5% down payment, and 1.5 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 1.5 points at closing?
   d. What are the monthly payments?
   e. How much interest is paid?

2. A couple wants to buy a house that costs $320,000. The house can be financed with a fixed rate mortgage of 15 years at 4.8% compounded monthly. They decide to make a 25% down payment, and 2.5 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 2.5 points at closing?
   d. What are the monthly payments?
   e. How much interest is paid?

3. A doctor wants to buy a house that costs $580,000. The house can be financed with a fixed rate mortgage of 15 years at 4.25% compounded monthly.
   She decides to make a 30% down payment, and 2 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 2 points at closing?
   d. What are the monthly payments?
   e. How much interest is paid?

4. A bachelor wants to buy an apartment that costs $110,000. The apartment can be financed with a fixed rate mortgage of 30 years at 5.75% compounded monthly. He decides to make a 5% down payment, and 1.5 points must be paid to the bank at closing.
   a. What is the down payment?
   b. What is the amount borrowed?
   c. What is the price of the 2.5 points at closing?

   d. What are the monthly payments?
   e. How much interest is paid?

5. A couple is trying to decide between two mortgage options to finance the cost of a $325,000 house. Find the monthly payments and the total interest paid for both options.
   a. The couple pays a 5% down payment and finances the house with a 30 year mortgage at 5.75% interest. No points are paid at closing.
   b. The couple pays a 25% down payment and finances the house with a 10 year mortgage at 5.25% interest. No points are paid at closing.
   c. Which option do you think is better for the couple?

6. A lawyer is trying to decide between two mortgage options to finance the cost of a $625,000 house. Find the monthly payments and the total interest paid for both options.
   a. The lawyer pays a 5% down payment and finances the house with a 30 year mortgage at 6.75% interest. No points are paid at closing.
   b. The lawyer pays a 30% down payment and finances the house with a 15 year mortgage at 6.25% interest. No points are paid at closing.
   c. Which option do you think is better for the lawyer?

7. A bachelor is trying to decide between two mortgage options to finance the cost of a $225,000 house. Find the monthly payments and the total interest paid for both options.
   a. The bachelor pays no down payment and finances the house with a 40 year mortgage at 6.25% interest. No points are paid at closing.
   b. The bachelor pays a 20% down payment and finances the house with a 15 year mortgage at 5.50% interest. No points are paid at closing.
   c. Which option do you think is better for the bachelor?

8. A couple is trying to decide between two mortgage options to finance the cost of a $350,000 house. Find the monthly payments and the total interest paid for both options.
   a. The couple pays a 5% down payment and finances the house with a 40 year mortgage at 5.9% interest. No points are paid at closing.
   b. The couple pays a 25% down payment and finances the house with a 10 year mortgage at 5.2% interest. No points are paid at closing.
   c. Which option do you think is better for the couple?

9. A fisherman wants to buy a top of the line boat that costs $40,000. To help him pay for it, the boat company offers to let him pay a 15% down payment with monthly payments at 9% interest compounded monthly for 10 years. What are the monthly payments? How much interest is paid to finance the boat?

10. To buy a car rather than paying cash the buyer considers financing it. For a $45,000 convertible the car company offers the individual a no down payment option with 5 year financing at a 7% interest rate. What are the monthly payments and what is the total interest paid to finance the purchase of the car?

11. To buy a car, rather than paying cash the buyer considers financing it. For a $37,500 convertible the car company offers the individual a 20% down payment option with 3 year financing at a 3% interest rate. What are the monthly payments and what is the total interest paid to finance the purchase of the car?

12. A student lost his funding for college and decided to finance his last two years largely by taking on credit card debt. When the student finally got his degree, he found out that he owed $65,000 to various credit cards. To pay off his debts the student destroyed his credit cards and consolidated his credit card debt. He took out a 10 year loan at 11.5% compounded monthly. How much are his monthly payments and what is the total interest that needs to be paid?

13. To pay off hospital bills a worker with poor credit has to take out a 10 year loan. If the worker borrows $60,000 at 9.5% compounded monthly, what will be the monthly payments and the total interest paid for the loan?

14. A student that goes to college finishes with $115,000 debt. The student’s loan is for 15 years at 8.5% interest compounded monthly. What are the monthly payments and what is the total interest paid for the loan?

7

Understanding how luck works goes a long way towards understanding the random events in the universe. Instead of being completely random it turns out that random events follow rules of probability, which depend on the conditions under which the events occur. Whether you are trying to win big in Las Vegas or simply count how many ways certain situations can occur understanding the basics is the beginning.

Course Outcomes:

- Calculate probabilities, and use and apply the normal distribution (Note that the normal distribution will be covered in Chapter 8.)

7.1 Counting Rules

The fundamental counting rule, permutations, and combinations are covered. Successful students will be able to determine the appropriate counting rule and calculate the result by hand or using a calculator.

7.2 Probabilities

The types of probabilities (subjective, empirical, and theoretical) are introduced. Theoretical probabilities and associated definitions are studied in depth. Contingency tables are shown.

7.3 Complement Rule and Addition Rule

Students need to focus on important aspects of the question to determine the appropriate probability rule. Mutually exclusive events are defined and used to determine the appropriate addition rule formula. Contingency tables are further developed as pertains to the addition rule.

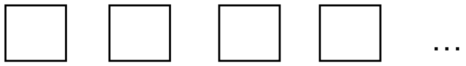
7.4 Multiplication Rule and Conditional Probabilities

Two more probability rules are studied. Independent and dependent events are defined and used to determine how to apply the multiplication rule. Conditional probabilities are explained by restricting the sample space according to the given condition. Contingency tables are reviewed and further developed as pertains to conditional probabilities.

Have you ever wondered how many different lottery tickets there are? Or, how many ways there are to pick people from a group to do a job? We often study these types of counting situations before we study probabilities. Here we will look at three counting rules and the good news is that we will always be asking, “How many …? or how many ways …?”

Fundamental Counting Rule

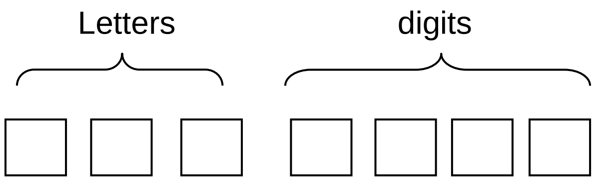
The number of ways that a sequence (or list) can occur is equal to the product of the number of ways that each element in the sequence can occur.



$$k_1 \cdot k_2 \cdot k_3 \cdot k_4 \cdot \ldots$$

Examples:

1. How many different license plates are there with three letters followed by four digits?



$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

There are 26 letters and 10 digits (0,1,2,3,4,5,6,7,8,9)

So, there are $26^3 \cdot 10^4 = 175{,}760{,}000$ different license plates.

Looking at the last four digits helps us to see where the counting rule comes from. Four digits could be

0000 , 0001 , 0002 , 0003, … , 9997 , 9998 , 9999

That would make 9999 + 1 (because of 0000) or 10,000 possibilities, which is the same as $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10{,}000$

2. How many different ways are there to have three children if we are considering whether the children are boys or girls?

$$\begin{array}{ccccccc} \square & & \square & & \square & & \\ 2 & \cdot & 2 & \cdot & 2 & = & 8 \end{array}$$

There are 8 possible boy/girl possibilities for 3 children

3. A new car can be bought with three types of options – color, motor, level of luxury. If there are five color options, three motor options, and two levels of luxury (basic or deluxe), how many different types of cars are there with these options?

$$\begin{array}{ccccccc} \text{color} & & \text{motor} & & \text{luxury} & & \\ \square & & \square & & \square & & \\ 5 & \cdot & 3 & \cdot & 2 & = & 30 \end{array}$$

There are 30 ways to buy the car.

Factorials

Before talking about the next two counting rules, it will be useful to understand factorials. For a whole number n, n factorial written $\mathrm{n}!$ means:

$\mathrm{n}! = \mathrm{n} \cdot (\mathrm{n}-1) \cdot (\mathrm{n}-2) \cdot \ldots \cdot 1$

So,

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$1! = 1$

$0! = 1$ This one is a bit strange.

Examples:

4. $\frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 = 210$

If we are doing factorials, cancelling common factors may be helpful.

5. $\frac{800!}{799!} = \frac{800 \cdot \cancel{799} \cdot \cancel{798} \cdot \cancel{797} \cdot \cancel{796} \cdot \cancel{795} \cdot \ldots}{\cancel{799} \cdot \cancel{798} \cdot \cancel{797} \cdot \cancel{796} \cdot \cancel{795} \cdot \ldots} = 800$

Here we really need to cancel the factors because $800!$ and $799!$ are too large to fit on our calculators. Basic scientific calculators do include a factorial button, !, that will calculate the factorials for us.

Permutations:

A permutation is a subset of a larger set where order matters. When we say order matters what we really want to say is that changing the members of the subset gives us a different situation. For instance horses coming in first, second, or third in a race of 10 horses is a permutation because changing which horse comes in first, second, and third gives us a different situation. Another example of a permutation is selecting president and vice president from a board of 15 members. Switching who is the president and who is the vice president gives another way to select the president and vice president.

To calculate the number of permutations we can use a formula or a scientific calculator. Using a scientific calculator to do the complete calculation is easier.

Formula for Permutation:

$$_{n}P_{k} = \frac{n!}{(n-k)!}$$

n = the size of the larger set from which smaller set is chosen
k = the size of the smaller subset

Examples:

6. How many ways can swimmers come in first, second, and third in a race of 12 swimmers?

Steps	Reasons
$_{12}P_{3}$	Permutation: there is a subset of three racers that are taken from the twelve racers and the order matters.
$_{12}P_{3} = \frac{12!}{(12-3)!}$	Formula: $_{n}P_{k} = \frac{n!}{(n-k)!}$
$= \frac{12!}{9!}$ $= \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot\ldots}{9\cdot8\cdot7\cdot\ldots}$ $= 12 \cdot 11 \cdot 10$	Calculate using the formula or skip this step by using a scientific calculator.

There are 1320 ways swimmers can come in first, second, and third in a race.

7. How many ways to choose a President, Vice President, Secretary, and Treasurer from a class of twenty students?

Steps	Reasons
$_{20}\mathrm{P}_4$	Permutation: there is a subset of four officers that are taken from the twenty students and the order matters because they have different jobs.
$_{20}\mathrm{P}_4 = \dfrac{20!}{(20-4)!}$	Formula: $_{\mathrm{n}}\mathrm{P}_{\mathrm{k}} = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{k})!}$
$= \frac{20!}{16!}$ $= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot \ldots}{16 \cdot 15 \cdot 14 \cdot \ldots}$ $= 20 \cdot 19 \cdot 18 \cdot 17$	Calculate using the formula or skip this step by using a scientific calculator.

There are 116,280 ways to choose a President, Vice President, Secretary, and Treasurer from a class of twenty students.

8. A lottery has one $50 prize and another $10 prize. If fifty people each buy one ticket, how many ways are there for the prizes to be awarded?

Steps	Reasons
$_{50}\mathrm{P}_2$	Permutation: there is a subset of two prize winners that are taken from the fifty people that buy tickets. Order matters because the prize values are different.
$_{50}\mathrm{P}_2 = \dfrac{50!}{(50-2)!}$	Formula: $_{\mathrm{n}}\mathrm{P}_{\mathrm{k}} = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{k})!}$
$= \frac{50!}{48!}$ $= \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot \ldots}{48 \cdot 47 \cdot 46 \cdot \ldots}$ $= 50 \cdot 49$	Calculate using the formula or skip this step by using a scientific calculator.

There are 2450 ways to award the two prizes.

Combinations:

A combination is a subset of a larger set where order does not matter. When we say “order does not matter” what we really want to say is that changing the members of the subset gives us the same situation. For instance selecting a committee of five members from a group of forty senators is a combination because we are taking a subset of five from the larger group of forty and the order is not important because the committee members cannot be distinguished in any way.

Formula for Combination:

$${}_{n}C_{k} = \frac{n!}{(n-k)!\,k!}$$

n = the size of the larger set from which smaller set is chosen
k = the size of the smaller subset

To calculate the number of combinations we can use a formula or a scientific calculator. Using a scientific calculator to do the complete calculation is easier.

Examples:

9. From the explanation of combinations above, how many are there to select a committee of five members from a group of forty senators?

Steps	Reasons
${}_{40}C_{5}$	Combination: there is a subset of five members on the committee and order does not matter because they have the same job.
${}_{40}C_{5} = \frac{40!}{(40-5)!\,5!}$	Formula: ${}_{n}C_{k} = \frac{n!}{(n-k)!k!}$
$= \frac{40!}{35!5!}$ $= \frac{40\cdot39\cdot38\cdot37\cdot36\cdot35\cdot34\cdot33\cdot\ldots}{(35\cdot34\cdot33\cdot\ldots)(5\cdot4\cdot3\cdot2\cdot1)}$ $= \frac{40\cdot39\cdot38\cdot37\cdot36}{5\cdot4\cdot3\cdot2\cdot1}$ $= 658{,}008$	Calculate using the formula or skip this step by using a scientific calculator.

There are 658,008 ways to select a committee of 5 out of 40 senators.

10. How many different five-card poker hands are there in a deck of fifty-two cards?

Steps	Reasons
${}_{52}C_{5}$	Combination: there is a subset of five cards and the order that the cards are dealt does not matter. We only need to know the final cards we hold and we may very well move them around in our hand.
$2{,}598{,}960$	Use a calculator with a combination button.

There are 2,598,960 different poker hands.

11. How many different possible samples of 10 students from a group of 200 students?

Steps	Reasons
$_{200}\mathrm{C}_{10}$	Combination: there is a subset of 10 students and the order that we select the students does not matter. The students do not have any jobs that will distinguish them from each other. If we list the names of the sample of ten students, the order that we write their names does not change anything.
$2.2451\ x\ 10^{16}$	Use a calculator with a combination button.

There are 22,451,004,300,000,000 different possible samples!

12. How many ways to choose 3 carpenters and 4 painters out of 7 carpenters and 12 painters?

Here we have something a little different. We have the combinations of carpenters and the combinations of painters. We will use the Fundamental Counting Rule to multiply the two combinations.

number of ways to choose the carpenters	number of ways to choose the painters

$$_{7}\mathrm{C}_{3} \cdot {}_{12}\mathrm{C}_{4} = 35 \cdot 495 = 17{,}325$$

There are 17,325 ways.

Exercises

Evaluate the following factorial problems:

1. $4!$

2. $7!$

3. $0!$

4. $5!$

5. $\frac{14!}{12!}$

6. $\frac{20!}{16!}$

7. $\frac{800!}{798!}$

8. $\frac{2000!}{1999!}$

9. $\frac{8!}{6!\cdot 2!}$

10. $\frac{10!}{3!\cdot 7!}$

11. $\frac{20!}{7!\cdot 3!}$

12. $\frac{15!}{4!\cdot 11!}$

Calculate the following using the Fundamental Counting Rule:

13. How many license plates are there that have two letters followed by five digits? Write two possible license plates.

14. How many license plates are there that have three letters followed by three digits? Write two possible license plates.

15. A diner has several breakfast options. A customer has three drink options (coffee, tea, or juice), five breakfast options (pancakes, waffles, omelet, fruit plate, or cereal), and three seating options (booth, table, or counter). How many different ways are there for a customer to enjoy breakfast? Write two possible breakfast options.

16. A diner has several lunch options. A customer has four drink options (coffee, tea, soda, or water), six food options (sandwich, salad, soup, meat, chicken, or fish), and three seating options (booth, table, or counter). How many different ways are there for a customer to enjoy lunch? Write two possible lunch options.

Calculate the following permutations. Using a calculator with a permutation button is easier than using the formula.

17. How many ways can horses come in first and second in a race with ten horses?

18. How many ways can sprinters come in first, second, and third in a race with ten sprinters?

19. How many ways are there for a class of 20 students to elect a president, vice president, secretary, and treasurer?

20. How many ways are there for a club with 15 members to elect a president, vice president, and treasurer?

Calculate the following combinations. Using a calculator with a combination button is easier than using the formula.

21. How many ways are there for a group of 18 students to select a committee of 4 members?

22. How many ways are there for a group of 10 professors to select a committee of 5 members?

23. From a group of 25 students how many ways are there to select a sample of 6 students to gain detailed feedback about a class?

24. From a group of 20 employees how many ways are there to select a sample of 8 employees to gain detailed feedback about working conditions?

25. How many ways to choose 2 carpenters and 5 painters out of 10 carpenters and 8 painters to renovate a house?

26. How many ways to choose 2 plumbers and 3 electricians out of 4 plumbers and 5 electricians to redo the wiring and plumbing in an old warehouse?

Calculate the following using the appropriate counting rule. Using a calculator with permutation and combination buttons where appropriate is easier than using the formulas.

27. An animal shelter has a variety of different dogs. A dog could be male or female. It could be large, medium, or small. The dog could be a pure breed or mixed. It could be a puppy, middle aged, or an older dog. How many different types of dogs can be adopted from the animal shelter if all of the above options are available?

28. From a class of 15 students, the professor asks 4 students for help moving the desks. How many ways are there for the professor to ask the students for help?

29. How many different ways are there to have five children if we are considering whether the children are boys or girls?

30. How many different ways are there to have six children if we are considering whether the children are boys or girls?

31. From a class of 15 students, the professor asks 4 students for help. One student will move desks, another student will take out the trash, a third student will clean out the bathrooms, and the fourth student will turn out the lights. How many ways are there for the professor to ask the students for help?

32. Five students decide to go see a concert. When they get back from the concert on Tuesday, they realize that they missed their math test. They go to the instructor and ask for a retest explaining that they had a flat tire and the spare was flat as well. When they open their make-up test, they see that there is only one question which is ‘Which tire was flat?’ How many ways could the five students answer which tire was flat?

33. How many ways are there to choose a President, Vice President, Secretary, and Treasurer from a class with 30 students?

34. A music club offers four free downloads as a promotion. If there are thirty songs to choose from, how many ways are there to download the four free promotional songs?

35. A chess club has a raffle to raise money. First prize is a bicycle, second prize is a chess set, and third prize is a $20 gift certificate. If 200 tickets are sold, how many ways are there for the prizes to be awarded?

36. To win the grand prize in a lottery somebody needs to pick five different numbers 1 to 60 as well as get the correct "series." The numbers cannot be repeated and there are 20 possibilities for the series. How many different lottery possibilities are there?

37. A sewing club has a raffle to raise money. Three brand new sewing machines are to be awarded. If 150 tickets are sold, how many ways are there for the prizes to be awarded?

38. In a math department there are 9 women and 6 men. How many ways are there to select 4 women and 2 men to represent the department at a conference?

39. In an English department there are 8 women and 15 men. How many ways are there to select 3 women and 5 men to represent the department at a conference?

40. A business man has three suits, five shirts, four ties, and two pairs of shoes. How many different ways can the businessman dress for a conference?

41. A beach lover has eight tee-shirts, four bathing suits, and three pairs of sandals. How many ways can he dress to go to the beach?

42. In order to make a fruit smoothie there are nine different types of fruit available. How many ways are there to choose four different types of fruit to make the smoothie?

43. A child wants to order a triple scoop ice cream cone at an ice cream shop. If there are twenty different types of ice cream to choose from and the child cares about the order in which the ice cream is stacked, how many different ways are there for the child to order the triple scoop ice cream cone?

44. From a group of thirty candidates, how many ways are there to select four of them to become Master Chief?

45. There are one hundred people that are given an experimental medicine. How many ways are there to select a sample of fifteen of them for further testing?

46. There are fifty entrees for an event in the Olympics. How many ways can the gold, silver, and bronze medals be awarded if it is impossible that two entrees receive the same medal?

47. A school lottery has one $500 prize and another $100 prize. If eighty people buy one ticket, how many ways are there for the two prizes to be awarded?

48. There are 7 boys and 10 girls in a high school class. How many ways are there to choose 2 boys and 3 girls to represent the class?

49. There are 15 doctors and 9 lawyers at a professional meeting on malpractice. How many ways are there to select 2 doctors and 3 lawyers to present at the meeting?

Probabilities are the likelihood of the occurrence of events. An outcome is an individual result of a probability experiment. An event is a collection of one or more outcomes. For instance if a fair die is rolled, there will be equal chances of rolling a 1, 2, 3, 4, 5, or 6. The 1, 2, 3, 4, 5, or 6 are the outcomes. The list of all possible outcomes is called the sample space. If we want to find the probability of rolling an odd number, then rolling an odd number is the event. The probability of rolling an odd number is $\frac{3}{6}$ or $\frac{1}{2}$.

There are three notions of probabilities, but we will only study the theoretical probabilities in depth.

1. Subjective probabilities refer to hunches or opinions about the likelihood of an event occurring. For example, a student may think there is an 80% chance that he will have fun in an economics class.

2. Empirical (or observational) probabilities say that the probability that an event will occur is the number of times that the event was observed divided by the total number of observations. For instance if I flip a coin 100 times and get heads 60 times, empirical probabilities say that the chance of getting heads when flipping the coin is $\frac{60}{100}$ or 60%.

3. Theoretical Probabilities depend on specific assumptions, which then will allow us to apply rules. Given equally likely outcomes, the probability of an event occurring is equal to the number of ways the event can occur divided by the total number of possible outcomes. For example if I flip a fair coin, the probability of getting heads is $\frac{1}{2}$ because there is one outcome of head out of two possible outcomes of head or tail.

We will only study theoretical probabilities.

Basic Probability Rule:

Assume that all possible outcomes are equally likely. Then the probability of an event E occurring is:

$$\mathrm{P(E)} = \frac{\text{number of ways that E can occur}}{\text{total number of possible outcomes}}$$

The symbols "$\mathrm{P(E)}$" mean “the probability of event E occurring.”

Examples:

1. For a-e, consider a jar with 10 tokens as follows:

5 red tokens numbered 1, 2, 3, 4, and 5
3 green tokens numbered 1, 2, and 3
2 yellow tokens numbered 1 and 2

All the tokens have the same chance of being selected. **One token is randomly selected.**

a. What is the probability of selecting a red token?

There are 5 red tokens.

$$\mathrm{P(red\ token)} = \frac{\text{number of ways that red can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(red\ token)} = \frac{5}{10} \text{ or } \frac{1}{2} \text{ or } .5 \text{ or } 50\%$$

b. What is the probability selecting a number 2 token?

There are 3 tokens with number 2, one for each color.

$$\mathrm{P(number\ 2)} = \frac{\text{number of ways that number 2 can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(number\ 2)} = \frac{3}{10} \text{ or } .3 \text{ or } 30\%$$

c. What is the probability of a token greater than 3?

There are 2 tokens with number greater than 3. The red 4 and red 5 are greater than 3.

$$\mathrm{P(number\ greater\ than\ 3)} = \frac{\text{number of ways that greater than 3 can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(number\ greater\ than\ 3)} = \frac{2}{10} \text{ or } \frac{1}{5} \text{ or } .2 \text{ or } 20\%$$

d. What is the probability of a yellow token?

There are 2 yellow tokens.

$$\mathrm{P(yellow\ tokens)} = \frac{\text{number of ways that greater than 3 can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(yellow\ tokens)} = \frac{2}{10} \text{ or } \frac{1}{5} \text{ or } .2 \text{ or } 20\%$$

e. What is the probability of a token that has a number of 6 or more?

There are no tokens with a number of 6 or more.

$$\mathrm{P(number\ 6\ or\ more)} = \frac{\text{number of ways that 6 or more can occur}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(number\ 6\ or\ more)} = \frac{0}{10} \text{ or } 0 \text{ or } 0\%$$

Notice that when an event cannot occur, it has a probability of 0. The maximum probability is 1 or 100%. In the example above, all ten numbered tokens form the sample space, which is the list of all possible outcomes. Each individual token is an outcome or individual result. The questions are about events.

2. For a-c, consider all of the equally likely outcomes for flipping a fair coin four times. The sample space (list of all possible outcomes is the following):

HHHH	HHHT	HHTH	HHTT	HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT	TTHH	TTHT	TTTH	TTTT

HHTH means heads on the first, second, and fourth tosses and tails on the third toss.

a. Find the probability of getting exactly one head.

Exactly one head can occur by HTTT, THTT, TTHT, or TTTH.

$$\text{P(exactly one head)} = \frac{\text{number of ways that exactly one head can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(exactly one head)} = \frac{4}{16} \ \text{ or } \ \frac{1}{4} \ \text{ or } .25 \ \text{ or } \ 25\%$$

b. Find the probability of getting at most 3 heads.

Remember that at most three means three or less. At most three heads can occur in any of the following ways:

	HHHT	HHTH	HHTT	HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT	TTHH	TTHT	TTTH	TTTT

$$\text{P(at most three heads)} = \frac{\text{number of ways that at most 3 heads can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(at most three heads)} = \frac{15}{16} \ \text{ or } \ .9375 \ \text{ or } \ 93.75\%$$

c. Find the probability of getting exactly two tails.

Exactly two tails can occur in the following six ways:

			HHTT		HTHT	HTTH	
	THHT	THTH		TTHH			

$$\text{P(exactly two tails)} = \frac{\text{number of ways that exactly two tails can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(exactly two tails)} = \frac{6}{16} \ \text{ or } \ .375 \ \text{ or } \ 37.5\%$$

Another type of problem we can do involves a chart. We will start by using the same Basic Probability Rule as above.

3. For a-c, learners were classified according to motivation and performance. Motivation could be low, average, or high. Performance could be poor or good. The results are in the chart:

Motivation

		Low	Average	High
Performance	Poor	14	8	3
	Good	2	6	7

a. One learner is randomly selected from this group of learners. Find the probability that the learner has good performance.

Whenever we have one of these chart problems the first thing we do is extend the chart by including totals.

Motivation

		Low	Average	High	Totals
Performance	Poor	14	8	3	25
	Good	2	6	7	15
	Totals	16	14	10	40

$$\text{P(learner has good performance)} = \frac{\text{number of ways that good performance can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(learner has good performance)} = \frac{15}{40} \text{ or } \frac{3}{8} \text{ or } .375 \text{ or } 37.5\%$$

b. One learner is randomly selected from this group of learners. Find the probability that the learner has high motivation.

$$\text{P(learner has high motivation)} = \frac{\text{number of ways that high motivation can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(learner has high motivation)} = \frac{10}{40} \text{ or } \frac{1}{4} \text{ or } .25 \text{ or } 25\%$$

c. One learner is randomly selected from this group of learners. Find the probability that the learner has average motivation and good performance.

$$\text{P(average motiv. and good perf.)} = \frac{\text{number of ways that average motiv. and good perf. can occur}}{\text{total number of possible outcomes}}$$

$$\text{P(average motiv. and good perf.)} = \frac{6}{40} \text{ or } \frac{3}{20} \text{ or } .15 \text{ or } 15\%$$

Exercises

Calculate the following probabilities:

Consider a six sided die with six equally likely outcomes: 1,2,3,4,5,6

1. What is the probability of rolling an even number?
2. What is the probability of rolling an odd number?
3. What is the probability of rolling less than 5?
4. What is the probability of rolling more than 5?
5. What is the probability of rolling at least 5?
6. What is the probability of rolling at most 5?
7. What is the probability of rolling less than 1?
8. What is the probability of rolling more than 6?

Consider all of the equally likely outcomes for flipping a fair coin three times where H represents a head and T represents a tail. The order represents the order of the flips. So, THH means that the first flip is tails while the second and third flips are heads. The sample space (list of all possible outcomes) is the following:

HHH HHT HTH HTT THH THT TTH TTT

9. What is the probability of getting exactly two heads?
10. What is the probability of getting exactly three tails?
11. What is the probability of getting at least two heads?
12. What is the probability of getting more than two tails?
13. What is the probability of getting exactly zero heads?
14. What is the probability of getting exactly one tail?
15. What is the probability of getting at most two tails?

16. What is the probability of getting at least one tail?

17. What is the probability of getting more than four tails?

18. What is the probability of getting less than zero heads?

Consider all of the equally likely outcomes for a family that has four children where B represents a boy and G represents a girl. The order represents the position in the family. So, BGGG means that the first child is a boy and the second, third, and fourth children are girls. The sample space (list of all possible outcomes is the following):

GGGG GGGB GGBG GGBB GBGG GBGB GBBG GBBB
BGGG BGGB BGBG BGBB BBGG BBGB BBBG BBBB

19. What is the probability of the family having exactly two boys?

20. What is the probability of the family having at least one girl?

21. What is the probability of the family having at most two girls?

22. What is the probability of the family having exactly three girls?

23. What is the probability of the family having exactly one boy?

24. What is the probability of the family having at most one girl?

25. What is the probability of the family having at least three boys?

26. What is the probability of the family having less than four girls?

27. What is the probability of the family having at least two girls?

28. What is the probability of the family having more than five girls?

29. What is the probability of the family having less than zero boys?

Consider a jar with 10 tokens as follows:

4 orange tokens numbered 1, 2, 3, and 4
3 blue tokens numbered 1, 2, and 3
3 red tokens numbered 1, 2, and 3

30. What is the probability of selecting a blue token?

31. What is the probability of selecting a number 1 token?

32. What is the probability of a token greater than 3?

33. What is the probability of a red token?

34. What is the probability of selecting an even numbered token?

Consider a jar with 15 tokens as follows:
6 yellow tokens numbered 1, 2, 3, 4, 5, and 6
3 blue tokens numbered 1, 2, and 3
6 green tokens numbered 1, 2, 3, 4, 5, and 6

35. What is the probability of selecting an even numbered token?

36. What is the probability selecting a number 6 token?

37. What is the probability of a token greater than 2?

38. What is the probability of a yellow token?

39. What is the probability of a blue token?

For the following, consider all the possible outcomes for rolling a fair die twice. The first number represents the first roll and the second number represents the second roll. For example 1,3 means that the first roll is 1 and the second roll is 3.

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

40. What is the probability that the sum of the two rolls is 4?

41. What is the probability that the sum of the two rolls is even?

42. What is the probability that the sum of the two rolls is at least 9?

43. What is the probability that the sum of the two rolls is less than 8?

44. What is the probability that the sum of the two rolls is 9?

45. What is the probability that the sum of the two rolls is at most 5?

46. What is the probability that the sum of the two rolls is more than 5?

47. What is the probability that the sum of the two rolls is more than 12?

48. What is the probability that the sum of the two rolls is at most 12?

Students are categorized by their attitude towards algebra and their ability to do statistics in the chart below. The students were asked if they like or dislike algebra and statistics ability was measured by their performance in an introductory statistics class. The results are in the chart:

		Statistics Ability		
		Low	Average	High
Algebra Attitude	Likes	5	12	10
	Dislikes	7	20	6

One student is randomly selected:

49. What is the probability that a randomly selected student dislikes algebra?

50. What is the probability that a randomly selected student has high statistics ability?

51. What is the probability that a randomly selected student likes algebra?

52. What is the probability that a randomly selected student has average statistics ability?

53. What is the probability that a randomly selected student has high statistics ability and likes algebra?

54. What is the probability that a randomly selected student has low statistics ability and likes algebra?

55. What is the probability that a randomly selected student likes algebra and has average statistics ability?

Dog owners that own only one dog are categorized by the size of the dogs that they have and whether the owners are female, male, or a couple. The results are in the chart:

		Dog Size		
		Small	Medium	Large
	Couple	15	14	9
Owners	Female	11	12	18
	Male	5	10	6

One dog is randomly selected:

56. What is the probability that the owner of a randomly selected dog is a female?

57. What is the probability that the owners of a randomly selected dog are a couple?

58. What is the probability that a randomly selected dog is large?

59. What is the probability that a randomly selected dog is small?

60. What is the probability of randomly selecting a small dog that is owned by a male?

61. What is the probability of randomly selecting a medium size dog that is owned by a couple?

Once we have the basic notion of probabilities, we can start applying various rules. The complement of an event is all other possible outcomes. For example the complement of winning the lottery is not winning the lottery. Likewise, the complement of not winning the lottery is winning the lottery. If we know the probability of an event, we may be able to guess at the probability of the complement. For instance if there is 10% chance of winning a lottery, then there is a 90% chance of not winning the lottery.

Complement Rule:

$P(\overline{E}) = 1 - P(E)$

$P(E) = 1 - P(\overline{E})$

$\overline{E}$ means the complement of E. We can also say “$\text{not } E$” instead of $\overline{E}$.

Examples:

1. If there is a .01 or 1% probability of winning a lottery, what is the probability of not winning the lottery?

Steps	Reasons
$P(\overline{E}) = 1 - P(E)$	E is the probability of winning the lottery.
$P(\overline{E}) = 1 - .01$	$\overline{E}$ is the probability of not winning the lottery.
$P(\overline{E}) = .99$	

There is a .99 or 99% probability of not winning the lottery.

2. There is a $\frac{1}{16}$ chance that a family with four children will have all girls. What is the probability of a family with four children having at least one boy?

Remembering that the complement of at least one is none will help in recognizing when to use the complement rule. After all if there is not at least 1 boy then they must be all girls. At least 1 boy could happen many ways, but not at least 1 boy can only happen in 1 way. So, the complement rule is quite useful.

Steps	Reasons
$P(E) = 1 - P(\overline{E})$	Let E be the event of at least 1 boy. Not at least 1 boy is all girls. So $\overline{E}$ is the event of all four girls
$P(\text{at least 1 boy}) = 1 - P(\text{four girls})$	
$P(\overline{E}) = 1 - \frac{1}{16}$	
$P(\overline{E}) = \frac{15}{16}$	

There is a $\frac{15}{16}$ or $.9375$ probability of having at least one boy.

Knowing that the complement of at least one is none will help in recognizing when to use the complement rule. Often students find applying the complement rule to be common sense. If there is a 20% chance of rain, then there is an 80% chance of it not raining, which is $1 - .2 = .8 \text{ or } 80\%$.

Addition Rule:

Look for:

1 item is selected; the word “or” is used; are the events mutually exclusive

Formula:

$\mathrm{P(A\ or\ B) = P(A) + P(B)}$ for mutually exclusive events A and B

$\mathrm{P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)}$ for not mutually exclusive events A and B

Mutually exclusive refers to events that cannot occur at the same time.

Examples:

3. For a-b, consider a jar with 10 tokens as follows:

    5 red tokens numbered 1, 2, 3, 4, and 5
    3 green tokens numbered 1, 2, and 3
    2 yellow tokens numbered 1 and 2

All the tokens have the same chance of being selected. **One token is randomly selected.**

a. What is the probability of selecting a red or green token?

Steps	Reasons
$\mathrm{P(red\ or\ green) = P(red) + P(green)}$ $\mathrm{P(red\ or\ green)} = \frac{5}{10} + \frac{3}{10}$ $\mathrm{P(red\ or\ green)} = \frac{8}{10}\ \mathrm{or}\ \frac{4}{5}\ \mathrm{or}\ .8$	One token is selected with “or”. Since there are no tokens that are both red and green the events are mutually exclusive. Use the addition rule for mutually exclusive.

b. What is the probability of selecting a red or number 2 token?

Steps	Reasons
$\mathrm{P(red\ or\ number\ 2) = P(red) + P(number\ 2) - P(red\ and\ number\ 2)}$	
$\mathrm{P(red\ or\ \#2)} = \frac{5}{10} + \frac{3}{10} - \frac{1}{10}$ $\mathrm{P(red\ or\ \#2)} = \frac{7}{10}\ \mathrm{or}\ .7$	One token is selected with “or”. Since there is a token that is both red and number 2 the events are not mutually exclusive. Use the addition rule for not mutually exclusive.

Here we see the reason for being careful about mutually exclusive. When we counted all the red tokens, we included the #2, red token. When we counted all #2 tokens, we include the red, #2 token. So the red, #2 token was counted twice. By subtracting it once we adjust for the double counting that occurs with not mutually exclusive.

4. In a *College Mathematics* class of the students that complete the course: 5 students get A's, 8 students get B's, 7 students get C's, 3 students get D's and 1 student gets an F. If one student is randomly selected what is the probability that the student gets an A or a B?

Steps	Reasons
$\mathrm{P(A\ or\ B)} = \mathrm{P(A)} + \mathrm{P(B)}$ $\mathrm{P(A\ or\ B)} = \frac{5}{24} + \frac{8}{24}$ $\mathrm{P(A\ or\ B)} = \frac{13}{24}$ or $.5417$	One student is selected with "or". Since no students earn grades of both A and B, the events are mutually exclusive. Use the addition rule for mutually exclusive. Count up all the grades to get the total number of students, which is 24. Round-off the answer to 3 or 4 decimal places.

5. Out of 200 cookies 120 contain chocolate, 60 contain walnuts, and 30 contain chocolate and walnuts. If one cookie is randomly selected, what is the probability that it contains chocolate or walnuts?

Steps	Reasons
$\mathrm{P(chocolate\ or\ walnut)} = \mathrm{P(chocolate)} + \mathrm{P(walnut)} - \mathrm{P(chocolate\ and\ walnut)}$	
$\mathrm{P(chocolate\ or\ walnut)} = \frac{120}{200} + \frac{60}{200} - \frac{30}{200}$	One cookie is selected. Since there are cookies that are both chocolate and walnut the events are not mutually exclusive. Use the addition rule for not mutually exclusive.
$\mathrm{P(chocolate\ or\ walnut)} = \frac{150}{200}$ or $\frac{3}{4}$ or $.75$	

Again notice that we had to subtract off the probability of selecting a cookie that was chocolate and walnut because those cookies were counted under chocolate and again under walnut.

6. For a-b, learners were classified according to motivation and performance. Motivation could be low, average, or high. Performance could be poor or good. The results are in the chart:

		Motivation		
		Low	Average	High
Performance	Poor	14	8	3
	Good	2	6	7

Whenever we have one of these chart problems the first thing we do is extend the chart by including totals.

		Motivation			
		Low	Average	High	Totals
Performance	Poor	14	8	3	25
	Good	2	6	7	15
	Totals	16	14	10	40

a. One learner is randomly selected from this group of learners. Find the probability that the learner has low or average motivation (first two columns).

Steps	Reasons
$\mathrm{P(low\ or\ average)} = \mathrm{P(low)} + \mathrm{P(average)}$ $\mathrm{P(low\ or\ average)} = \frac{16}{40} + \frac{14}{40}$ $\mathrm{P(low\ or\ average)} = \frac{30}{40}$ or $\frac{3}{4}$ or .75	One learner is selected with “or”. Since no learners have both low and average motivation, the events are mutually exclusive. Use the addition rule for mutually exclusive. There are 24 students.

b. One learner is randomly selected from this group of learners. Find the probability that the learner has average motivation or good performance.

Steps	Reasons
$\mathrm{P(average\ motiv.\ or\ good\ perf.)} = \mathrm{P(aver.\ motiv.)} + \mathrm{P(good\ perf.)} - \mathrm{P(aver\ mot.\ and\ good\ perf.)}$	
$\mathrm{P(aver\ mot\ or\ good\ perf)} = \frac{14}{40} + \frac{15}{40} - \frac{6}{40}$ $\mathrm{P(aver\ mot\ or\ good\ perf)} = \frac{23}{40}$ or .575	Whenever we have chart problems, a row and column will overlap. In this problem there are learners with average motivation and good performance. The events are not mutually exclusive. Use the addition rule for not mutually exclusive.

Looking at chart problems that use the addition rule, we need to watch for the following:

1. One item is being selected
2. The word “or” is used.

The first two are required for the addition rule.

3. Two columns never intersect. So, two columns are mutually exclusive.
4. A row and a column always intersect. So, a row and column are not mutually exclusive.

Exercises

Calculate the following probabilities:

1. If there is a .0001 or .01% probability of winning a lottery, what is the probability of not winning the lottery?

2. If there is a .0005 or .05% probability of winning a lottery, what is the probability of not winning the lottery?

3. There is a $\frac{1}{128}$ chance that a family with seven children will have all boys. What is the probability of a family with seven children having at least one girl?

4. There is a $\frac{1}{64}$ chance that a family with six children will have all girls. What is the probability of a family with six children having at least one boy?

There are 400 active duty military on a remote base. 175 are in the Army, 100 are in the Air Force, 75 are in the Navy, and 50 are Marines. One of the active duty military members is randomly selected.

5. What is the probability that the military member is in the Navy or Army?

6. What is the probability that the military member is in the Marines or Air Force?

7. What is the probability that the military member is in the Air Force or Navy?

8. What is the probability that the military member is in the Army or Marines?

Consider a jar with 10 tokens as follows:
  4 orange tokens numbered 1, 2, 3, and 4
  3 blue tokens numbered 1, 2, and 3
  3 red tokens numbered 1, 2, and 3
All the tokens have the same chance of being selected. One token is randomly selected.

9. What is the probability that an orange or blue token is selected?

10. What is the probability that a number 2 or number 3 token is selected?

11. What is the probability that a number 2 or blue token is selected?

12. What is the probability that a red or number 3 token is selected?

13. What is the probability that an orange or number 4 token is selected?

14. What is the probability that a blue or number 4 token is selected?

15. What is the probability that a number 1 or red token is selected?

16. What is the probability that an orange or number 3 token is selected?

17. What is the probability that a number 1 or number 2 token is selected?

Consider a jar with 15 tokens as follows:
7 yellow tokens numbered 1, 2, 3, 4, 5, 6, and 7
3 blue tokens numbered 1, 2, and 3
5 green tokens numbered 1, 2, 3, 4, and 5

18. What is the probability that a yellow or green token is selected?

19. What is the probability that a number 2 or number 5 token is selected?

20. What is the probability that a number 2 or green token is selected?

21. What is the probability that a yellow or number 3 token is selected?

22. What is the probability that a blue or number 5 token is selected?

23. What is the probability that a green or number 7 token is selected?

24. What is the probability that a number 1 or blue token is selected?

25. What is the probability that a blue or number 3 token is selected?

26. What is the probability that a number 1 or number 2 token is selected?

27. Out of 250 cookies 140 contain chocolate, 80 contain walnuts, and 40 contain chocolate and walnuts. If one cookie is randomly selected, what is the probability that it contains chocolate or walnuts? What is the probability that the cookie contains neither chocolate nor walnuts?

28. In a *College Mathematics* class of the students that complete the course: 6 students get A’s, 9 students get B’s, 8 students get C’s, 4 students get D’s and 2 students get F’s. If one student is randomly selected what is the probability that the student gets an A or a B? What is the probability that the student gets neither an A nor a B?

Students are categorized by their attitude towards algebra and their ability to do statistics in the chart below. The students were asked if they like or dislike algebra and statistics ability was measured by their performance in an introductory statistics class. The results are in the chart:

		Statistics Ability		
		Low	Average	High
Algebra	Likes	5	12	10
Attitude	Dislikes	7	20	6

One student is randomly selected:

29. What is the probability that a randomly selected student has low statistics ability or high statistics ability?

30. What is the probability that a randomly selected student has low statistics ability or average statistics ability?

31. What is the probability that a randomly selected student has average statistics ability or high statistics ability?

32. What is the probability that a randomly selected student dislikes algebra or has average statistics ability?

33. What is the probability that a randomly selected student likes algebra or has high statistics ability?

34. What is the probability that a randomly selected student likes algebra or has low statistics ability?

35. What is the probability that a randomly selected student dislikes algebra or has high statistics ability?

36. What is the probability that a randomly selected student has average statistics ability or likes algebra?

37. What is the probability that a randomly selected student has average statistics ability or dislikes algebra?

Dog owners that own only one dog are categorized by the size of the dogs that they have and whether the owners are female, male, or a couple. The results are in the chart:

			Dog Size	
		Small	Medium	Large
	Couple	15	14	9
Owners	Female	11	12	18
	Male	5	10	6

One dog is randomly selected:

38. What is the probability that the owner of a randomly selected dog is female or male?

39. What is the probability that the owners of a randomly selected dog are a couple or that the owner is male?

40. What is the probability that a randomly selected dog is small or large?

41. What is the probability that a randomly selected dog is small or medium?

42. What is the probability of randomly selecting a small dog or that the owner is a male?

43. What is the probability of randomly selecting a small dog or that the owner is a female?

44. What is the probability of randomly selecting a large dog or that the owner is a male?

45. What is the probability of randomly selecting a medium dog or that the owner is a male?

46. What is the probability of randomly selecting a medium dog or that the owners are a couple?

47. What is the probability of randomly selecting a small dog or that the owners are a couple?

48. What is the probability that the owners of a randomly selected dog are a couple or that the dog is small?

49. What is the probability that a randomly selected dog is medium or large?

When we select more than one item does the occurrence of the first event affect the probability of the second event? The answer is maybe. For example when playing cards if the cards are put back into the deck after they are drawn we still have the same 52 cards for later draws. However, if we are dealt five cards to play a card game, we are not replacing the cards and the later draws are affected by the previous draws because there are fewer cards. Here we are looking at the concept of independent and dependent events.

Events are independent if the occurrence of the first affect does not affect the probability of the occurrence of the second event. If we draw a card look at it, put the card back, then draw a second card the events of drawing the first and second card are independent. However, if the first card is not put back before drawing the second, then the events are dependent (not independent) because taking out a card affects the probability of drawing the second card. Events are dependent (not independent) if the occurrence of the first event does affect the probability of the second event.

Multiplication Rule:

Look for:

Two or more items are drawn.

The idea of "and", which can be expressed as all, both, each, none, etc. For example Jack and Fred go to the mall is the same as saying they both go to the mall.

Are the events independent?

Formula:

$\mathrm{P(A\ and\ B) = P(A) \cdot P(B)}$

We need to consider whether or not the probability of the second event B occurring is changed by the occurrence of the first event A. (independent vs. dependent)

Examples:

1. For a-d, consider a jar with 10 tokens as follows:

5 red tokens numbered 1, 2, 3, 4, and 5
3 green tokens numbered 1, 2, and 3
2 yellow tokens numbered 1 and 2

All the tokens have the same chance of being selected.

Remember basic probabilities for an individual event. For example,

$$\mathrm{P(drawing\ a\ red\ token)} = \frac{\text{number of ways that red can occur}}{\text{total number of possible outcomes}} = \frac{5}{10} = .5$$

Remember that "$\mathrm{P(dawing\ a\ red\ token)}$" means "the probability of drawing a red token."

a. Draw two tokens without replacing the first token before drawing the second token. What is the probability of drawing two red tokens?

Steps	Reasons
$\mathrm{P(A\ and\ B) = P(A) \cdot P(B)}$ $\mathrm{P(first\ red\ \ and\ second\ red) = P(1st\ red) \cdot P(2nd\ red)}$	Use the multiplication rule because two items are drawn and drawing two red tokens is the same as saying the first token is red and the second token is red.
$\mathrm{P(1st\ red\ and\ 2nd\ red)} = \frac{5}{10} \cdot \frac{4}{9}$	Notice that the probability of the second red is changed because the first token was not put back (dependent). There are one less red token for the numerator and one less token for the denominator.
$\mathrm{P(1st\ red\ and\ 2nd\ red)} = \frac{20}{90}\ \mathrm{or}\ \frac{2}{9}\ \mathrm{or}\ .2222$	Write answer as a reduced fraction or decimal number rounded-off to three or four places.

b. Draw one token, replace it, and draw a second token. What is the probability of drawing two red tokens?

Steps	Reasons
$\mathrm{P(A\ and\ B) = P(A) \cdot P(B)}$ $\mathrm{P(first\ red\ \ and\ second\ red) = P(1st\ red) \cdot P(2nd\ red)}$	Use the multiplication rule because two items are drawn and drawing two red tokens is the same as saying the first token is red and the second token is red.
$\mathrm{P(1st\ red\ and\ 2nd\ red)} = \frac{5}{10} \cdot \frac{5}{10}$	Notice that the probability of the second red is not changed because the first token was put back (independent). There are the same number of tokens for the denominator and the numerator.
$\mathrm{P(1st\ red\ and\ 2nd\ red)} = \frac{25}{100}\ \mathrm{or}\ \frac{1}{4}\ \mathrm{or}\ .25$	Write answer as a reduced fraction or decimal number.

Examples a. and b. show the importance of putting the token back (independent) and not putting the token back (dependent) when applying the multiplication rule.

c. Draw three tokens. Replace the token each time before drawing the next token. What is the probability of first drawing a red token, next drawing a green token and lastly drawing a yellow token?

Steps	Reasons
$\mathrm{P(1st\ red\ \ and\ 2nd\ green\ and\ 3rd\ yellow)} = \mathrm{P(1st\ red)} \cdot \mathrm{P(2nd\ green)} \cdot \mathrm{P(3rd\ yellow)}$	
$\mathrm{P(1st\ red\ \ and\ 2nd\ green\ and\ 3rd\ yellow)}$ $\frac{5}{10} \cdot \frac{3}{10} \cdot \frac{2}{10}$ $\frac{30}{1000}$ or $\frac{3}{100}$ or .03	Use the multiplication rule because three items are drawn with the idea of “and” used throughout. The first is red and the second is green and the third is yellow. Since the tokens are replaced the probabilities remain the same and the events are independent.

d. Draw three tokens in a row. What is the probability that the first two are number 2’s and last one is a number 1?

Steps	Reasons
Multiplication Rule $\mathrm{P(1st\ is\ \#2\ \ and\ 2nd\ is\ \#2\ and\ 3rd\ is\ \#1)}$	Two or more items are selected. We are using the idea of “and” to connect events: the first draw is a #2 and the second is #2 and the third is #1.
$\mathrm{P(1st\ \#2)} \cdot \mathrm{P(2nd\ \#2)} \cdot \mathrm{P(3rd\ \#1)}$ $\frac{3}{10} \cdot \frac{2}{9} \cdot \frac{3}{8}$ $\frac{18}{720}$ or $\frac{1}{40}$ or .025	When drawing three tokens in a row, the tokens are not replaced before each draw. So, these are dependent events. Notice that the denominators decrease for each drawn token that is left out. The numerator of the second draw decreases because there is one fewer number two token. For the third draw there are still 3 number one tokens.

2. A six-sided die is rolled twice. What is the probability of rolling a 2 followed by an odd number?

The six equally likely outcomes for rolling the die once are 1,2,3,4,5, and 6.

Steps	Reasons
$\text{P(A and B)} = \text{P(A)} \cdot \text{P(B)}$ $\text{P(first \#2 and second odd)} = \text{P(1st \#2)} \cdot \text{P(2nd odd)}$	Use the multiplication rule because the die is rolled twice and the first roll is a number 2 and the second roll is odd.
$\text{P(1st red and 2nd red)} = \frac{1}{6} \cdot \frac{3}{6}$	Notice that the probability of the second roll is not affected by the first roll because the die is not changed in any way (independent). There is one #2 out of six possible outcomes. There are three odd possibilities (1,3,5) out of six total possible outcomes.
$\text{P(1st red and 2nd red)} = \frac{3}{36} \text{ or } \frac{1}{12} \text{ or } .0833$	Write answer as a reduced fraction or decimal number.

Conditional probabilities can be thought of as an alteration of the original problem. When we have the multiplication rule for dependent events the probability of the second event really is a conditional probability. In the example above the probability of drawing the second red ball given that the first red ball was drawn and not replaced is $\frac{4}{9}$ instead of $\frac{5}{10}$. The trick to conditional probabilities is to restrict the problem to whatever comes after the given and ignore the rest of the information. There is also a formula.

Formula for Conditional Probabilities

$$\text{P(B given A)} = \frac{\text{number of outcomes common to B and A}}{\text{number of outcomes for A}}$$

If we look at the formula carefully, we are finding the probability of event B occurring restricted to the situation where the event A has occurred.

Rather than trying to apply the formula, the easiest way to do conditional probabilities will be to just look at the outcomes after the word “given”.

Examples:

3. For a-c, consider our jar with 10 tokens as follows:
    5 red tokens numbered 1, 2, 3, 4, and 5
    3 green tokens numbered 1, 2, and 3
    2 yellow tokens numbered 1 and 2
All the tokens have the same chance of being selected.

a. On one draw what is the probability of drawing a number 2 token given that a green token is drawn?

$\text{P}(\#2 \text{ drawn given a green token drawn})$

Since it is given that a green token is drawn we are only interested in the green tokens.
There are three green tokens numbered 1, 2, and 3.
There is one number 2 out of three green tokens. So,

$\text{P}(\#2 \text{ drawn given a green token drawn}) = \frac{1}{3} \text{ or } .3333$

b. On one draw what is the probability of drawing an odd number given that a red token is drawn?

$\text{P}(\text{odd drawn given a red token is drawn})$

Since it is given that a red token is drawn we are only interested in the red tokens.
There are five red tokens numbered 1, 2, 3, 4, and 5.
There are three odd red tokens (1,3,5) out of five red tokens. So,

$\text{P}(\text{odd drawn given a red token is drawn}) = \frac{3}{5} \text{ or } .6$

c. On one draw what the probability of drawing a green token given that a number 3 token is drawn.

$\text{P}(\text{green token given number 3 token is drawn})$

Since it is given that a number 3 token is drawn we are only interested in the number 3 tokens.
There are two #3 tokens: red #3 and green #3. So,

$\text{P}(\text{green token given number 3 token is drawn}) = \frac{1}{2} \text{ or } .5$

4. As before for a-b, learners were classified according to motivation and performance. Motivation could be low, average, or high. Performance could be poor or good. The results are in the chart:

		Motivation		
		Low	Average	High
Performance	Poor	14	8	3
	Good	2	6	7

Whenever we have one of these chart problems the first thing we do is extend the chart by including totals.

		Motivation			
		Low	Average	High	Totals
Performance	Poor	14	8	3	25
	Good	2	6	7	15
	Totals	16	14	10	40

a. One learner is randomly selected from this group of learners. Find the probability that the learner has low motivation given that he has good performance.

When we calculate conditional probabilities from a chart, we are only looking at a single row or a single column. When we say:
"probability that the learner has low motivation given that he has good performance"

The phrase after "given" tells us where to look. So, for good performance only (it is given that there is good performance), we look at the good performance row.

		Motivation			
		Low	Average	High	Totals
Performance	Poor				
	Good	2	6	7	15
	Totals				

The probability of low motivation for the above row of good performance is
$\frac{2}{15}$ or $.1333$

We can also use the formula for conditional probabilities with the chart:

$$\text{P(B given A)} = \frac{\text{number of outcomes common to B and A}}{\text{number of outcomes for A}}$$

$$\text{P(low motiv. given good performance)} = \frac{\text{number of good performance and low motivation}}{\text{number of outcomes for good performance}} = \frac{2}{15} \text{ or } .1333$$

b. One learner is randomly selected from this group of learners. Find the probability that the learner has poor performance given that he has average motivation.

Since we are given average motivation, we need look at only the average motivation column.

		Motivation		
		Low	Average	High
Performance	Poor		8	
	Good		6	
	Totals		14	

The probability of poor performance for the above column of average motivation is $\frac{8}{14}$ or $\frac{4}{7}$ or $.5714$

We can also use the formula for conditional probabilities with the original chart:

$$\text{P(B given A)} = \frac{\text{number of outcomes common to B and A}}{\text{number of outcomes for A}}$$

$$\text{P(poor perform. given average motiv.)} = \frac{\text{number of poor perform and average motivation}}{\text{number of outcomes for average motivation}}$$

$$= \frac{8}{14} \text{ or } \frac{4}{7} \text{ or } .5714$$

5. There are three switches each with a 20% chance of failing.

a. What is the probability that all three switches fail?

Steps	Reasons
$\text{P(three switches fail)} = \text{P(1st fail and 2nd fails and 3rd fails)}$	All three is the same as “and.”
$\text{P(1st fails)} \cdot \text{P(2nd fails)} \cdot \text{P(3rd fails)}$	Multiplication rule with three items selected with “and.”
$.2 \cdot .2 \cdot .2$	
$.008$	The probability of each switch failing is .2 and then multiply.

b. What is the probability that a single switch does not fail?

Steps	Reasons
$\text{P(a single switch does not fail)}$	“does not fail” is the complement of “does fail”
$\text{P(does not fail)} = 1 - \text{P(fails)}$	Complement rule
$\text{P(does not fail)} = 1 - .2 = .8$	

c. Of the three switches what is the probability that no switches fail?

Steps	Reasons
$\mathrm{P}(\text{no switches fail }) = \mathrm{P}(\text{1st not fail and 2nd not fail and 3rd not fail})$	No switches is the same as "and."
$\mathrm{P}(\text{1st not fail}) \cdot \mathrm{P}(\text{2nd not fail}) \cdot \mathrm{P}(\text{3rd not fail})$	Multiplication rule with three items selected with "and."
$.8 \cdot .8 \cdot .8$	The probability of each switch not failing is .8 and then we multiply.
$.512$	

d. What is the probability that at least one switch does not fail?

Steps	Reasons
$\mathrm{P}(\text{at least one does not fail })$	"at least one" can occur many ways: any one switch could fail, any two switches could fail, or any three switches could fail. In fact there are so many possibilities that it will be easier to use the compliment rule
$\mathrm{P}(\text{at least one fails}) = 1 - \mathrm{P}(\text{not at least one fails})$	Complement rule
$\mathrm{P}(\text{at least one fails}) = 1 - \mathrm{P}(\text{none fail})$	"not at least one" is the same as saying "none" That is the real trick. If we see "at least one", we may want to use the compliment rule.
$1 - .512$	The probability that none of the three switches fails is coming directly from step c. of this problem.
$.488$	

Exercises

Calculate the following:

A six-sided die with equally likely outcomes {1,2,3,4,5,6} is rolled twice. Find the probability of

1. First rolling an even number and then rolling an odd number.
2. First rolling a number 3 and then rolling an even number.
3. First rolling a number greater than 2 and then rolling an even number.
4. First rolling an odd number and then rolling a number 5.
5. First rolling a number that is at least 3 and then rolling a number that is at most 3.
6. First rolling a number less than 5 and then rolling a number more than 4

A six-sided die with equally likely outcomes {1,2,3,4,5,6} is rolled three times. Find the probability of

7. Rolling three even numbers.
8. Rolling three odd numbers.
9. Rolling an even number on the first roll, rolling at least a number 5 on the second roll, and rolling a number more than 2 on the third roll.
10. Rolling three numbers that are all at most 5.
11. Rolling three numbers that are all at least 3.
12. Rolling an odd number on the first roll, rolling at least a number 4 on the second roll, and rolling a number more than 2 on the third roll.

Consider a jar with 10 tokens as follows:
- 4 orange tokens numbered 1, 2, 3, and 4
- 3 blue tokens numbered 1, 2, and 3
- 3 red tokens numbered 1, 2, and 3

All the tokens have the same chance of being selected.

13. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are orange?

14. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are blue?

15. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are orange?

16. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are blue?

17. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 2 and the second token is a number 3?

18. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 1 and the second token is a number 4?

19. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is a number 3 and the second token is number 4?

20. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is number 1 and the second token is number 2?

21. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are blue?

22. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are orange?

23. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are blue?

24. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are orange?

25. One token is drawn. What is the probability that the token is a number 1 given that it is orange?

26. One token is drawn. What is the probability that the token is a number 2 given that it is blue?

27. One token is drawn. What is the probability that the token is a number 3 given that it is red?

28. One token is drawn. What is the probability that the token is an even number given that it is orange?

29. There are three switches each with a 30% chance of failing.
    a. What is the probability that all three switches fail?
    b. What is the probability that a single switch does not fail?
    c. Of the three switches what is the probability that no switches fail?
    d. Of the three switches what is the probability that at least one switch fails?

30. There are three switches each with a 25% chance of failing.
    a. What is the probability that all three switches fail?
    b. What is the probability that a single switch does not fail?
    c. Of the three switches what is the probability that no switches fail?
    d. Of the three switches what is the probability that at least one switch fails?

Consider a jar with 15 tokens as follows:
7 yellow tokens numbered 1, 2, 3, 4, 5, 6, and 7
3 blue tokens numbered 1, 2, and 3
5 green tokens numbered 1, 2, 3, 4, and 5
All the tokens have the same chance of being selected.

31. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are yellow?

32. Draw a token. Replace the token and then draw a second token. What is the probability that both tokens are green?

33. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are yellow?

34. Draw two tokens without replacing the first token before drawing the second. What is the probability that both tokens are green?

35. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 2 and the second token is a number 3?

36. Draw a token. Replace the token and then draw a second token. What is the probability that the first token is a number 2 and the second token is a number 5?

37. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is a number 3 and the second token is a number 6?

38. Draw two tokens without replacing the first before drawing the second. What is the probability that the first token is a number 1 and the second token is a number 2?

39. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are yellow?

40. Draw three tokens by replacing the tokens before drawing the next token. What is the probability that all three tokens are green?

41. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are yellow?

42. Draw three tokens without replacing the tokens before drawing the next token. What is the probability that all three tokens are green?

43. One token is drawn. What is the probability that the token is a number 1 given that it is yellow?

44. One token is drawn. What is the probability that the token is a number 2 given that it is blue?

45. One token is drawn. What is the probability that the token is a number 3 given that it is green?

46. One token is drawn. What is the probability that the token is an odd number given that it is yellow?

47. One token is drawn. What is the probability that the token is an even number given that it is yellow?

48. One token is drawn. What is the probability that the token is an odd number given that it is blue?

Students are categorized by their attitude towards algebra and their ability to do statistics in the chart below. The students were asked if they like or dislike algebra and statistics ability was measured by their performance in an introductory statistics class. The results are in the chart:

		Statistics Ability		
		Low	Average	High
Algebra Attitude	Likes	5	12	10
	Dislikes	7	20	6

One student is randomly selected:

49. What is the probability that a randomly selected student has low statistics ability?

50. What is the probability that a randomly selected student has high statistics ability?

51. What is the probability that a randomly selected student has average statistics ability given that the student likes algebra?

52. What is the probability that a randomly selected student has average statistics ability given that the student dislikes algebra?

53. What is the probability that a randomly selected student likes algebra and has high statistics ability?

54. What is the probability that a randomly selected student likes algebra and has low statistics ability?

55. What is the probability that a randomly selected student likes algebra or has high statistics ability?

56. What is the probability that a randomly selected student dislikes algebra or has low statistics ability?

57. What is the probability that a randomly selected student likes algebra given that the student has average statistics ability?

58. What is the probability that a randomly selected student dislikes algebra given that the student has average statistics ability?

Dog owners that own only one dog are categorized by the size of the dogs that they have and whether the owners are female, male, or a couple. The results are in the chart:

		Dog Size		
		Small	Medium	Large
Owners	Couple	15	14	9
	Female	11	12	18
	Male	5	10	6

One dog is randomly selected:

59. What is the probability that the owner of a randomly selected dog is a male?

60. What is the probability that a randomly selected dog is medium?

61. What is the probability of randomly selecting a medium dog that is owned by a male?

62. What is the probability of randomly selecting a large dog that is owned by a couple?

63. What is the probability of randomly selecting a medium dog given that is owned by a male?

64. What is the probability of randomly selecting a large dog given that is owned by a couple?

65. What is the probability of randomly selecting a small dog given that the owner is male?

66. What is the probability of randomly selecting a small dog given that the owner is female?

67. What is the probability of randomly selecting a medium dog or that the owner is a female?

68. What is the probability of randomly selecting a large dog or that the owner is a female?

69. What is the probability that the owners of a randomly selected dog are a couple given that the dog is medium?

70. What is the probability that the owner of a randomly selected dog is a male given that the dog is small?

8

Statistics are around us both seen and in ways that affect our lives without us knowing it. We have seen data organized into charts in magazines, books and newspapers. That's descriptive statistics! We can also measure the spread of the data set or the data set's most "typical" value. Comparing pairs of related values helps us in prediction. Understanding the type and strength of the relation between two different variables is the first step.

Course Outcomes:

- Efficiently use relevant technology
- Demonstrate proficiency in basic concepts and procedures related to descriptive statistics
- Calculate probabilities, and use and apply the normal distribution

8.1 Frequency Distribution, Frequency Polygon, Histogram

Data is organized into frequency distributions. Students learn to present the results of the frequency distributions as frequency polygons and histograms.

8.2 Measure of Central Tendency

Measures of central tendency all arrive at one value to represent the whole data set. Several different measures of central tendency are found. The most important measures of central tendency are mean, median, and mode.

8.3 Measures of Dispersion

The spread of the data is measured and represented with a single value. Standard deviation measures the spread of the data around the mean. Range is the difference between the largest and the smallest data values.

8.4 Percentiles and Normal Distribution

Percentiles are defined in order to use the standard normal distribution chart. Percentages and probabilities are found using the normal distribution.

8.5 Correlation

The relationship between two variables is examined. Students learn scatter plots and types of correlation – positive, negative, and no correlation. The strength of the correlation is introduced.

When we ask the question what are statistics, we may come up with various answers. One common notion is the various facts that are involved in a situation. For example, we may talk about the technical capabilities of a computer: amount of memory, speed of the central processing unit, power input, etc. Here we are really describing the computer.

There are two types of statistics:
1. Descriptive statistics are used to collect, organize and present data
2. Inferential statistics uses information about a sample to say something about a population.

The population is all the values of interest. A sample is a subset of the population. The connection between a sample and a population is based on probabilities. For example, we may look at how 100 students feel about the new educational resources to find out how all students feel about the new educational resources. In the class *Introductory Statistics* we study descriptive statistics, probabilities, and inferential statistics.

If we study 100 students that take MATH 103, the 100 students is the sample. All students taking MATH 103 is the population.
If observe 25 cars that come out of a factory. The population is all the cars that come out of the factory. The sample is the 25 cars that we observe.

Data refers to a set of observations or possible outcomes. There are two types of data:

1. Qualitative: non-numerical data which refers to a characteristic of the data being observed
2. Quantitative: numerical data, which can be of two types
   a. Quantitative Discrete: most likely the result of counting how many
   b. Quantitative Continuous: the data can take any of a arrange of values; for quantitative continuous for any two values there is always another possible value between them

Qualitative data includes:
Students' hair color, American's religious background, ethnic background of students taking a standardized test

Quantitative discrete data includes:
The number of students entering the tenth grade, the number of red cars, the number of high schools in a town

Quantitative continuous data includes:
The distance from home to work, the weight of candy bars
Note that for any two distances 5.2 km and 5.3 km there is always another value in between such as 5.25 km. The same is true for weight.

Here we are going to focus on descriptive statistics. If we have a list of data, we may want to start by organizing it into a frequency distribution. A frequency distribution contains a column of classes and their frequencies.

Consider the years of education of a group of young adults:

12	10	15	12	15	13	11	14	11	13
13	14	13	13	11	14	12	16	15	16

The frequency distribution is as follows:

Years of education	Number of young adults
10	1
11	3
12	3
13	5
14	3
15	3
16	2

The first column contains the classes and the second column contains the frequencies. The frequency distribution lets us interpret the data more easily:

- 10 years of education is the value that occurs the least (only once)
- 13 years of education is the most frequently occurring value (called the mode)
- The data is fairly equally spread around the value of 13 years of education, equally spread data is said to be symmetric.
- There are 8 young adults that have at least 14 years of education. At least 14 years is 14 years or more (14 years or 15 years or 16 years). 3+3+2=8 young adults

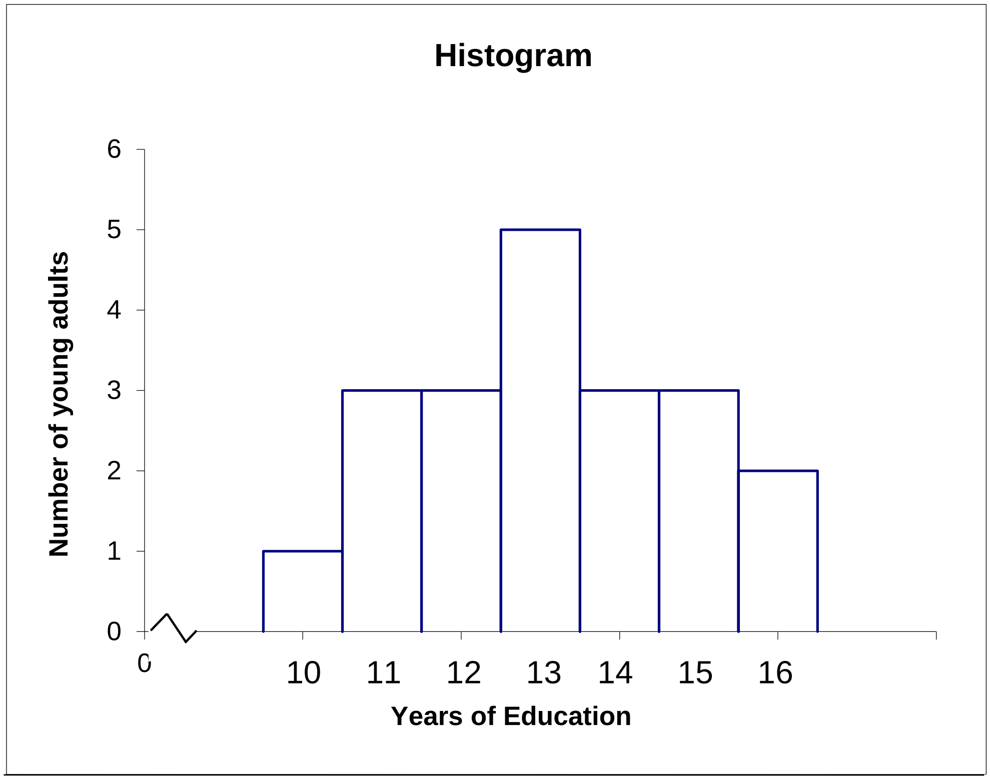
We create a frequency distribution by making classes starting with the smallest value and ending with the highest value. Then we tally all of the values to get the frequency for each class.

Some characteristics of frequency distributions:

- Classes are exhaustive – all values fall in some class
- Mutually exclusive classes – the classes do not overlap
- Avoid open-ended classes like more than 12 years
- 5 to 20 classes is typical for a frequency distribution
- Equal size classes

Histograms are bar charts where the vertical axis is for the frequency and the horizontal axis is for the classes. The bars will touch.

Using the frequency distribution from above:

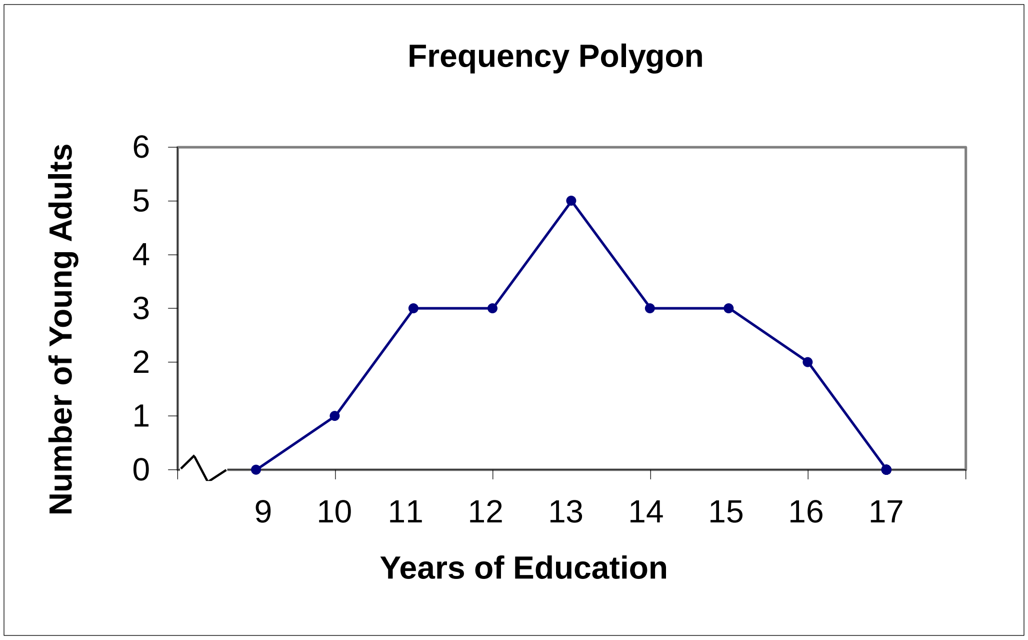


Here we have a histogram. The /\/ symbol on the horizontal axis indicates that we are skipping out to a higher value

A frequency polygon is a line graph with the vertical axis representing the frequencies and horizontal axis representing the class (or class midpoint). To make a polygon, we can draw the class prior to the frequency distribution, which is 9 and the class after the distribution, which is 17 with frequencies of zero.

The frequency distribution was as follows:

Years of education	Number of young adults
10	1
11	3
12	3
13	5
14	3
15	3
16	2



Both the histogram and the frequency polygon give us similar types of information. We see a picture of the shape of the graph. In both cases we can see that our data (years of education) is somewhat bell shaped.

Exercises

Say whether the following data is qualitative, quantitative discrete, or quantitative continuous.

1. The number of fish in an aquarium

2. Types of species found on a remote island

3. Speed of an Internet connection

4. The number of students in a statistics class

5. Color of a dog

6. Percent of alcohol in a popular adult beverage

7. Thirty families were asked how many electronic devices (tablets, cell phones, and computers) they have in their home. The number of devices for each family is listed below:

3	5	2	6	1	3	3	6	4	5
2	3	5	4	6	3	5	5	2	4
8	4	8	8	3	6	5	6	5	4

Make a frequency distribution.
How many families had at least 3 electronic devices in their home?

8. Forty office workers were asked how many cups of coffee that they drink in a day. The number of cups of coffee for each worker is listed below:

2	0	2	1	3	4	2	0	2	5
3	2	2	3	2	5	2	2	3	3
2	0	3	2	3	1	0	2	0	6
4	2	1	0	0	3	2	4	2	3

Make a frequency distribution.
How many of the office workers drink at most three cups of coffee in a day?

9. Twenty students were asked how many pairs of sunglasses that they lost over the last year. The number of sunglasses that each student lost is listed below:

1	0	2	1	0	0	1	0	3	0
0	1	2	0	1	3	1	0	2	1

Make a frequency distribution.
How many students did not lose any sunglasses over the last year?

10. The number of gas stations in eighteen towns was counted. The number of gas stations in the towns is listed below:

7	10	9	10	11	7	8	12	10
11	8	10	7	9	11	10	11	9

Make a frequency distribution.
How many of the towns have between 9 and 11 gas stations inclusive?

11. Make a histogram for number 7. Label the axes. How many families had at most 3 electronic devices in their home?

12. Make a histogram for number 8. Label the axes. How many of the office workers drink at least three cups of coffee in a day?

13. Make a histogram for number 9. Label the axes. How many students lost at least 1 pair of sunglasses over the last year?

14. Make a histogram for number 10. Label the axes. How many of the towns have between 7 and 9 gas stations inclusive?

15. Make a frequency polygon for number 7. Label the axes. How many families had between 2 and 4 electronic devices inclusive in their home?

16. Make a frequency polygon for number 8. Label the axes. How many of the office workers drink between 1 and 3 cups of coffee inclusive in a day?

17. Make a frequency polygon for number 9. Label the axes. How many students lost less than 4 pairs of sunglasses over the last year?

18. Make a frequency polygon for number 10. Label the axes. How many of the towns have at least 10 gas stations?

19. How are the frequency distribution, histogram, and frequency polygon similar? How are they different?

20. Do you prefer a histogram or a frequency polygon? Why?

For a data set, we may want to come up with one value to represent the whole data set. For instance a student over her college career will have various different grades for a multitude of courses. So, schools will assign 4.0 to an A, 3.0 to a B, 2.0 to a C, 1.0 to D, and 0.0 to a F and then take the average of all the student's grades on this 4, 3, 2, 1, 0 scale. The result is the grade point average (GPA). A student's GPA is a quick way to determine how a student did throughout her college career by looking at only one value.

Mean (also called arithmetic mean) for individual data:

$$\bar{x} = \frac{\sum x}{n}$$

$\bar{x}$ is the mean
Σ means add all the values; it is the Greek letter "sigma" which is used for sums
x are the individual data values
n is the number of data values

This notation is new for many of us but the idea of adding all the values and dividing by the number of values is the "average" that we have seen throughout our lives.

Example:

1. Five people consume the following number of calories in a day:

2000	2500	2000	3200	2300

Find the mean.

$$\bar{x} = \frac{\sum x}{n} = \frac{2000+2500+2000+3200+2300}{5} = \frac{12,000}{5} = 2400 \text{ calories}$$

The mode is the most frequently occurring value. Another non-statistical way to think of mode is as the prevailing fashion. That may help us remember that mode is the most popular or most frequently occurring data value.

Example:

2. Five people consume the following number of calories in a day:

2000	2500	2000	3200	2300

Find the mode.

The mode is 2000 calories because it occurs most often.

The median is another measure of central tendency. When the data is ordered small to large (or large to small) the median is the data value in the middle. For an even number of data values the median may lie between two data values in which case we take the mean of those two data values.

Examples:

3. Five people consume the following number of calories in a day:

2000 2500 2000 3200 2300

Find the median.

First order the data small to large:

2000 2000 2300 2500 3200

The median is the value in the middle.

The median is 2300 calories.

4. Six students graduate from college at the following ages:

22 30 27 42 22 47

Find the median.

First order the data small to large:

22 22 27 30 42 47

The median is the value in the middle.

Since the median is between two data values, we take the mean of the 27 and 30. $\frac{27+30}{2} = 28.5$. The median age for graduation is 28.5 years old.

The midrange is the data value in the middle of the lowest and highest values. midrange = $\frac{\text{lowest data value + highest data value}}{2}$ We can also say the midrange is the mean of the lowest and highest data values.

Example:

5. Five people consume the following number of calories in a day:

2000 2500 2000 3200 2300

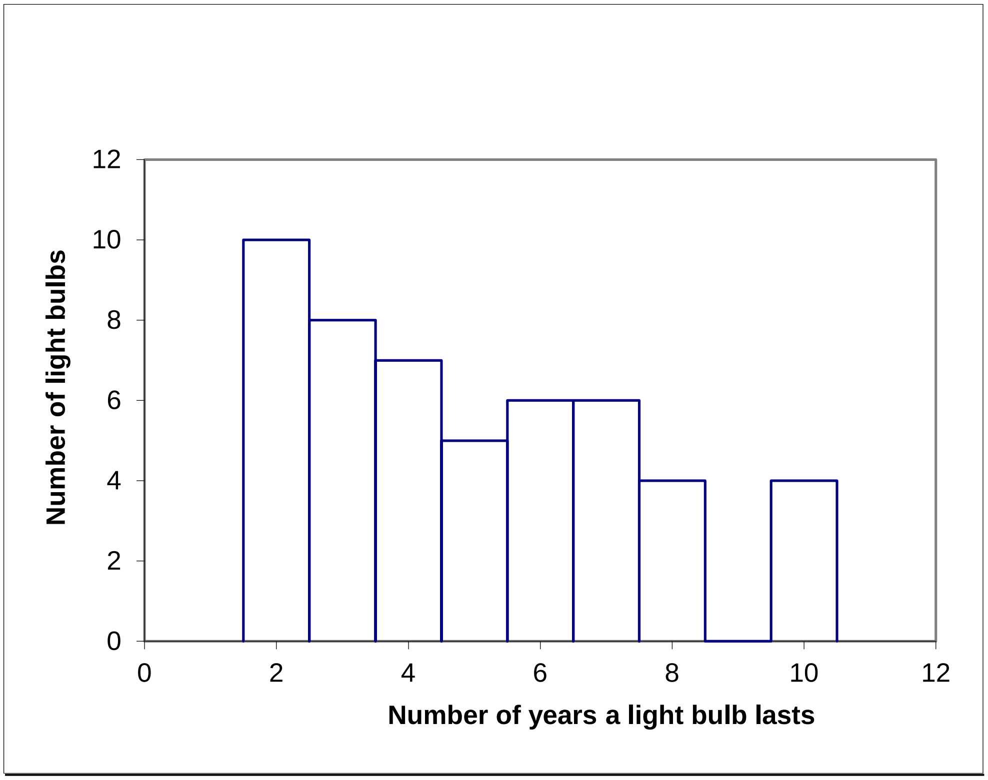
Find the midrange.

midrange = $\frac{\text{lowest data value + highest data value}}{2} = \frac{2000+3200}{2} = 2600$ calories

All four measures of central tendency will have the same units as the original, individual data values. However, we may get four different values for the four measures of central tendency (mean, mode, median, midrange), as in the calorie example above. So, which measure of central tendency is the best? That will certainly depend. The midrange and the mean are affected by extreme values. In fact the midrange is only dependent on the lowest and highest values, which may not be representative. The mode is only dependent on the one value that is repeated the most. So, the mode is easy to calculate, but it does not take into account all of the data values. The median is the middle of the data when it is ordered. So, extreme values do not affect it. The mean turns out to be very important in large part because all of the data values are used in its calculation.

We can also calculate the measures of central tendency from data that is grouped into a frequency distribution, histogram, or frequency polygon. A good first step will often be to write the histograms or frequency polygons as a frequency distribution.

Consider the actual life span of 50 light bulbs:



We can translate the histogram into a frequency distribution. Remember the classes are along the horizontal axis and the frequencies are along the vertical axis.

Lifespan of light bulb in years	Number of light bulbs
2	10
3	8
4	7
5	5
6	6
7	6
8	4
9	0
10	4

The mode from a frequency distribution is easy. The mode is the data value with the highest frequency. In the light bulb example the mode is a lifespan of 2 years because it has the highest frequency, which is 10. For a histogram or frequency polygon it is the class with the highest bar or point.

The midrange is just the $\frac{\text{lowest data value + highest data value}}{2} = \frac{2+10}{2} = 6$ years. Since the grouped data displays lowest and highest values we can just look at the chart or graph to get the needed values.

The mean gets a little more complicated. To find the mean we add up all the data values. Since there are 10 light bulbs with a 2 year lifespan we would be adding 2+2+2+2+2+2+2+2+2+2=20, but it is much easier to just multiply the data value by the frequency 2·10=20. The same holds true for all the other classes. There are 8 light bulbs with a 3 year life span. Rather than adding 3+3+3+3+3+3+3+3=24, it is easier to multiply data value by frequency 3·8=24.
By grouped data we mean a frequency distribution, frequency polygon, or histogram.

Formula for mean of grouped data:

$$\bar{x} = \frac{\sum f \cdot x_m}{n}$$

$\bar{x}$ is the mean
Σ means add all the values. It is called a summation.
f is the frequency for each class
x_m= class (or class midpoint)
n is the number of data values, which is found by adding up all of the frequencies

Use the frequency distribution to make a table:

x_m	f	$f \cdot x_m$
2	10	20
3	8	24
4	7	28
5	5	25
6	6	36
7	6	42
8	4	32
9	0	0
10	4	40
	50	247

The sum of the frequencies is total number of data values n.

Adding the $f \cdot x_m$ gives us the sum of all the individual data values or $\sum f \cdot x_m$.

The mean is

$$\bar{x} = \frac{\sum f \cdot x_m}{n} = \frac{247}{50} = 4.94 \text{ years}$$

Finally, we have the median. Remember the median is the middle of the data listed from low to high. It is a little tricky at first because we tend to overlook that there are 10 values of 2 years, 8 values of 3 years, 7 values of 4 years, so on and so forth.

2 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 4 4 4 4 4 4 4 5 5 5 5 5 6 6 6 6 6 6 7 7 7 7 7 7 8 8 8 8 10 10 10 10

The median is between 4 and 5.
So, we say the median is 4.5 years

x_m	f	Cumulative frequency
2	10	10
3	8	18
4	7	25
5	5	30
6	6	
7	6	
8	4	
9	0	
10	4	

There are 25 data values in the first three classes, which is halfway to the total of 50 data values. The median must be between 4 years and 5 years, which is 4.5 years.

These four measures of central tendency can be calculated from a list of individual data or when the data is grouped into a frequency distribution, frequency polygon, or histogram. The above light bulb example shows how to find the measures of central tendency for a histogram or frequency distribution. To find the measures of central tendency for a frequency polygon is similar.

Exercises

A group of five students received the following grades on a quiz:

100 70 90 80 95

1. Find the mean.
2. Find the mode.
3. Find the median.
4. Find the midrange.

A group of seven students received the following grades on a quiz:

70 85 70 100 95 75 65

5. Find the mean.
6. Find the mode.
7. Find the median.
8. Find the midrange.
9. Which do you think is the best measure of central tendency and why?

A class of thirty students had the following number of absences per week:

0 5 2 6 0 3 4 1

10. Find the mean.
11. Find the mode.

12. Find the median.

13. Find the midrange.

Instructors are evaluated on a five point scale for many different attributes and then given an overall score. The overall score for a group of ten instructors is listed below:

4.5 3.9 4.4 4.5 3.7 4.8 4.7 4.5 3.5 4.1

14. Find the mean.

15. Find the mode.

16. Find the median.

17. Find the midrange.

18. Which do you think is the best measure of central tendency and why?

In summertime children often get lost on the beach and then the parents have to go pick the children up at the Beach Public Services Office. The number of children taken to the Beach Public Services Office for ten days is listed below:

7 4 6 6 2 5 6 3 2 3

19. Find the mean.

20. Find the mode.

21. Find the median.

22. Find the midrange.

At a supermarket when people drop a box of eggs on the floor and make a mess they usually do not pay for the eggs or pick them up. Over a week the number of cartons of broken eggs can mount up. At a large supermarket the store manager decided to keep track of the number of cartons of eggs that were dropped and cleaned up on a weekly basis. Over twelve weeks the number of cartons of eggs dropped and then cleaned up by store employees is listed below:

11 5 15 17 12 15 3 8 19 15 7 10

23. Find the mean.

24. Find the mode.

25. Find the median.

26. Find the midrange.

Forty office workers were asked how many cups of coffee that they drink in a day with the following frequency distribution:

Cups of Coffee	Number of workers
0	7
1	3
2	15
3	9
4	3
5	2
6	1

27. Find the mean.

28. Find the mode.

29. Find the median.

30. Find the midrange.

31. Which do you think is the best measure of central tendency and why?

Twenty students were asked how many pairs of sunglasses that they lost over the last year. The number of sunglasses that each student lost is listed below:

Pairs of Sunglasses	Number of students
0	8
1	7
2	3
3	2

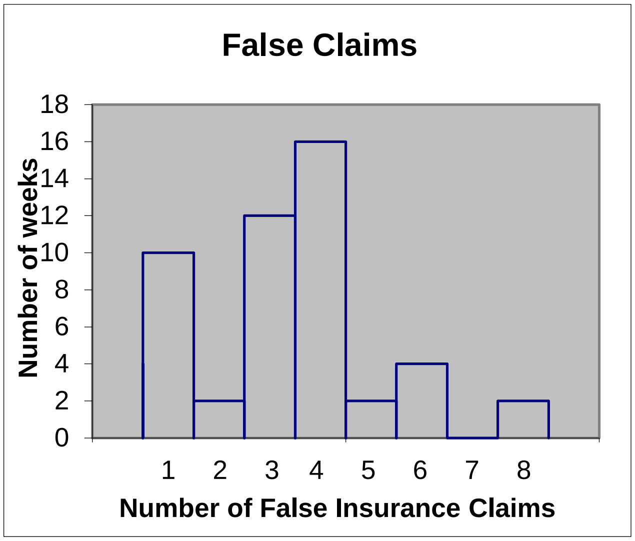
32. Find the mean.

33. Find the mode.

34. Find the median.

35. Find the midrange.

An insurance company receives a lot of false claims over a year. The number of false claims is counted and organized into the following histogram:



36. Find the mean.

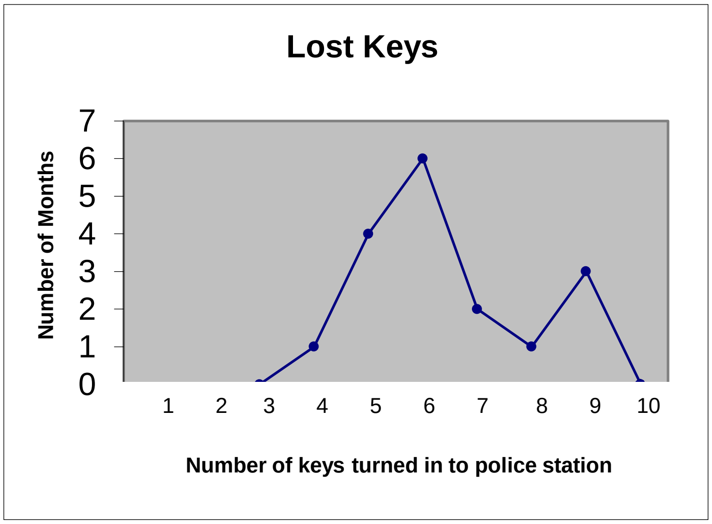
37. Find the mode.

38. Find the median.

39. Find the midrange.

40. Which do you think is the best measure of central tendency and why?

When people lose their keys, the keys are often turned into the local police station. Consider the following frequency polygon for the number of keys turned into local police station per month:



41. Find the mean.

42. Find the mode.

43. Find the median.

44. Find the midrange.

In the last section we saw how to find one data value to represent data, but we may also want to know whether the data is close together or spread out. Consider the following two sets of data:

Data Set 1:

20	20	20	50	80	80	80

Data Set 2:

40	40	40	50	60	60	60

The mean, median, and midrange are 50 for both data sets, but the first data set is more spread out in than the second. We will be concerned with two measure of dispersion – the range and the standard deviation.

The range is the highest value minus the lowest value. The range for Data Set 1 is 80 – 20 = 60. The range for Data Set 2 is 60 – 40 = 20.

To measure the spread of the data (or dispersion), we might try to compare each data value to the mean by subtracting the mean from each data value as in the table below. Remember $\bar{x}$ = mean and x= data values .
For Data Set 1:

x	$x-\bar{x}$	
20	-30	for 20 – 50
20	-30	for 20 – 50
20	-30	for 20 – 50
50	0	for 50 – 50
80	30	for 80 – 50
80	30	for 80 – 50
80	30	for 80 – 50

Here we have compared each data value to the mean, but when we add the column $x-\bar{x}$ the sum is zero and it will be zero every time. There are two choices at this point. Firstly, we could take the absolute value of the $x-\bar{x}$ column. Then all the values would be positive and we could find some average distance to the mean, which is called the mean deviation:

x	$x-\bar{x}$	$\lvert x-\bar{x}\rvert$
20	-30	30
20	-30	30
20	-30	30
50	0	0
80	30	30
80	30	30
80	30	30
		180

$$\sum |x-\bar{x}| = 180$$

$$\text{Mean Deviation} = \frac{\sum|x-\bar{x}|}{n} = \frac{180}{7} = 25.71$$

This mean deviation is an average distance from the mean for all the data values. It is easy enough to understand, but it turns out that another measure of dispersion is far more important. Let’s consider the data values minus the mean again, but this time we will square those differences instead of taking the absolute value.

x	$x - \bar{x}$	$(x - \bar{x})^2$
20	-30	900
20	-30	900
20	-30	900
50	0	0
80	30	900
80	30	900
80	30	900
		5400

Not to worry too much about why, but squaring the $x - \bar{x}$ is another way to get positive values that will not cancel when we add them.

Formula for standard deviation:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

s = standard deviation
x =data values
$\bar{x}$ = mean
$\sum$ = add all the values
n = the number of data values

The units for standard deviation are the same as the units for the original data values.

The standard deviation measure the spread of the data around the mean.

From our above chart for Data Set 1, we have $\sum(x - \bar{x})^2 = 5400$. Because there are 7 data values $n = 7$. So, the standard deviation for Data Set 1 is

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{\frac{5400}{7-1}} = 30$$

Example:

Calculate the standard deviation for the following ages of students

25 21 37 18 32 27 45 19

X	$\mathrm{x}-\bar{\mathrm{x}}$	$(\mathrm{x}-\bar{\mathrm{x}})^2$
25	-3	9
21	-7	49
37	9	81
18	-10	100
32	4	16
27	-1	1
45	17	289
19	-9	81
224		626

First find the mean:

$\bar{\mathrm{x}} = \frac{\sum \mathrm{x}}{\mathrm{n}} = \frac{224}{8} = 28$ years old

To find the standard deviation:

Subtract the mean from each data value, square those differences, and then take the sum of those squared differences:

$s = \sqrt{\frac{\sum(\mathrm{x}-\bar{\mathrm{x}})^2}{\mathrm{n}-1}} = \sqrt{\frac{626}{8-1}} = 9.46$ years

Finding the standard deviation with the formula has some disadvantages:

1. If the mean is a rounded-off decimal value, then there will be a lot of rounding error in the final answer.
2. If we have a frequency distribution, frequency polygon, or histogram, then it may be very difficult to calculate the standard deviation. Imagine that there are several hundred data values that are grouped into a histogram.

The good news is that a modern scientific calculator will let you calculate the mean and standard deviation of individual data (a list of values) or grouped data (frequency distribution, frequency polygon, or histogram). We just load the data into the calculator and it does all the work for us.

So, what does the standard deviation measure?

The standard deviation measures the spread of the data around the mean.

Exercises

A group of five students received the following grades on a quiz:

100 70 90 80 95

1. Find the range.
2. Find the standard deviation.

A group of seven students received the following grades on a quiz:

70 85 70 100 95 75 65

3. Find the range.
4. Find the standard deviation.

Consider the following two sets of test scores:
Set 1:
60 60 80 100 100

Set 2:
60 75 80 85 100

5. Find both means.
6. Find both ranges.
7. Find both standard deviations.
8. Do the means and ranges indicate that there is a difference in the two data sets?
9. What do the standard deviations show about the two data sets?

A class of thirty students had the following number of absences per week:

0 5 2 6 0 3 4 1

10. Find the range.

11. Find the standard deviation.

Instructors are evaluated on a five point scale for many different attributes and then given an overall score. The overall score for a group of ten instructors is listed below:

4.5 3.9 4.4 4.5 3.7 4.8 4.7 4.5 3.5 4.1

12. Find the range.

13. Find the standard deviation.

In summertime children often get lost on the beach and then the parents have to go pick the children up at the Beach Public Services Office. The number of children taken to the Beach Public Services Office for ten days is listed below:

7 4 6 6 2 5 6 3 2 3

14. Find the range.

15. Find the standard deviation.

16. What does the standard deviation measure?

At a supermarket when people drop a box of eggs on the floor and make a mess they usually do not pay for the eggs or pick them up. Over a week the number of cartons of broken eggs can mount up. At a large supermarket the store manager decided to keep track of the number of cartons of eggs that were dropped and cleaned up on a weekly basis. Over twelve weeks the number of cartons of eggs dropped and then cleaned up by store employees is listed below:

11 5 15 17 12 15 3 8 19 15 7 10

17. Find the range.

18. Find the standard deviation.

Forty office workers were asked how many cups of coffee that they drink in a day with the following frequency distribution:

Cups of Coffee	Number of workers
0	7
1	3
2	15
3	9
4	3
5	2
6	1

19. Find the standard deviation.

20. Find the range.

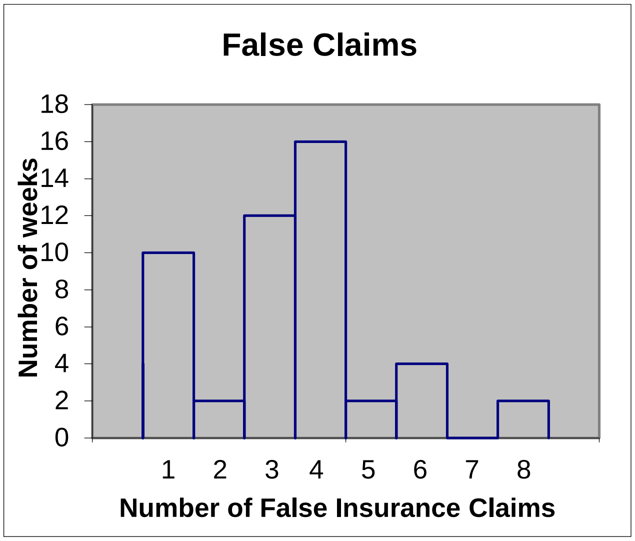
Twenty students were asked how many pairs of sunglasses that they lost over the last year. The number of sunglasses that each student lost is listed below:

Pairs of Sunglasses	Number of students
0	8
1	7
2	3
3	2

21. Find the standard deviation.

22. Find the range.

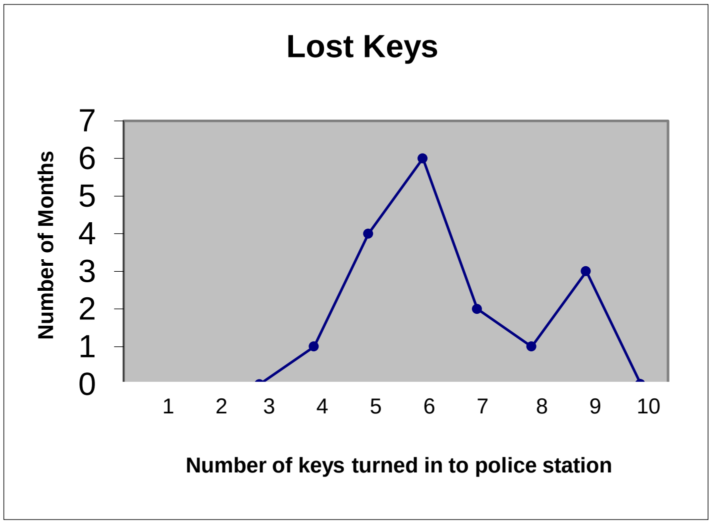
An insurance company receives a lot of false claims over a year. The number of false claims is counted and organized into the following histogram:



23. Find the standard deviation.

24. Find the range.

When people lose their keys, the keys are often turned into the local police station. Consider the following frequency polygon for the number of keys turned into local police station per month:



25. Find the standard deviation.

26. Find the range.

Standardized tests such as Navy Advancement Exams, Scholarship Aptitude Test (SAT), and IQ tests may be based on scores of 80, 1600, 100 or some other score. These scores may or may not mean much too most people depending on the individual's experience with the specific exam. So, along with these various scores, the results will often include a percentile score.

Percentile refers to the value that is above a given percent of the data. For example a score of 700 on the Math SATs may put an individual in the 93rd percentile. That means that the score of 700 is higher than 93% of the individuals that took the test.

Percentiles are also used when talking about peoples physical characteristics like height and weight. A height of 57 inches is the 81st percentile for a 10 year old boy, which means a 57 inch tall 10 year old boy is taller than 81% of all 10 year old boys.

The normal distribution, which is bell-shaped distribution, is found throughout nature. Characteristics of people like intelligence, height, and weight follow the normal distribution. The normal distribution is based on the normal curve.

Characteristics of the normal curve:
1. bell-shaped
2. symmetric
3. area under the curve is 1 or 100%
4. mean, mode, median are in the middle
5. the curve approaches the horizontal axis
6. the shape is completely determined by the mean and standard deviation
7. the empirical rule applies

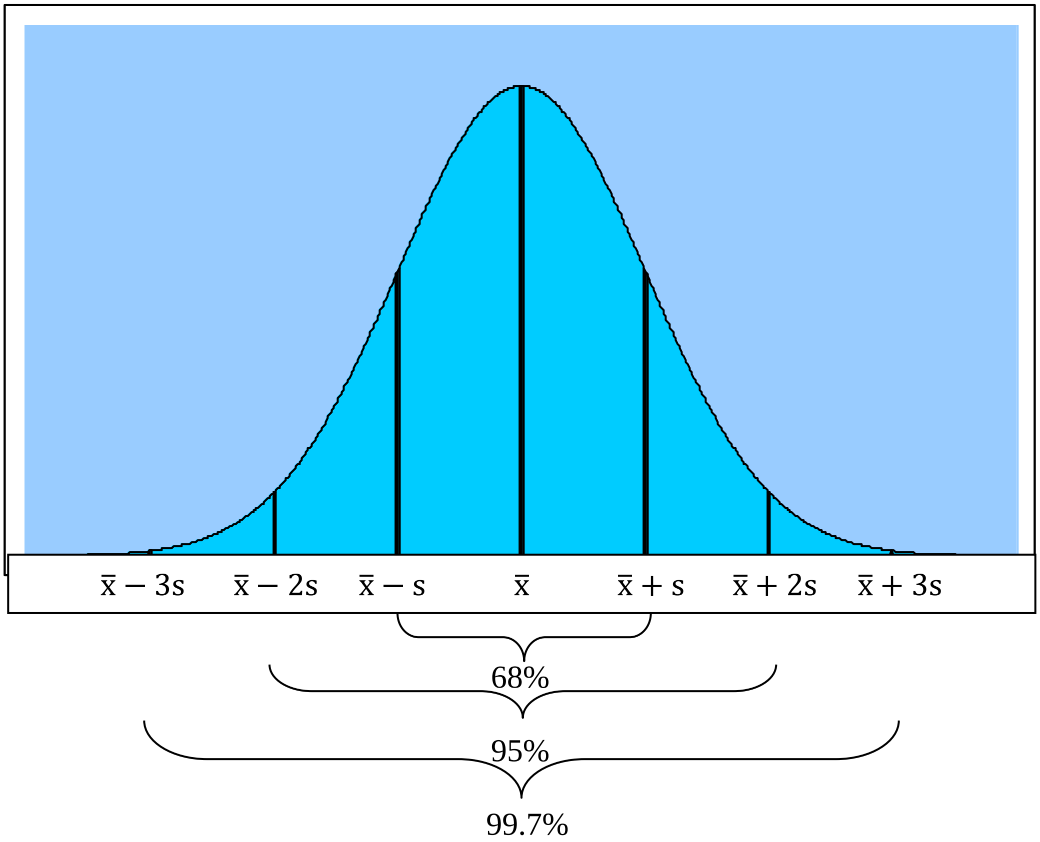
The empirical rule states that for normally distributed data:

68% of the data is within 1 standard deviation of the mean
95% of the data is within 2 standard deviations of the mean
99.7% of the data is within 3 standard deviation of the mean

Below is a picture of the empirical rule and normal curve. Remember that

$\bar{x}$ is the mean
s is the standard deviation
$\bar{x} - 2s$ is the mean minus to standard deviations. So, between $\bar{x} - 2s$ and $\bar{x} + 2s$ is within 2 standard deviations of the mean.

Notice the shape of the normal curve as well as the empirical rule.

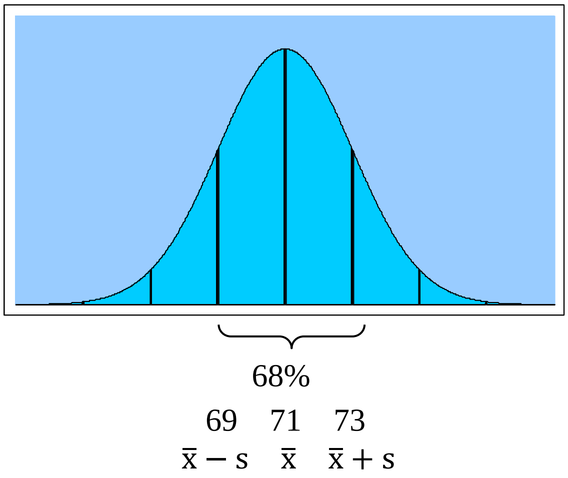


Examples:

1. Heights for adult males are normally distributed with a mean of 71 inches and a standard deviation of 2 inches.

a. What percentage of adult males are between 69 and 73 inches tall?

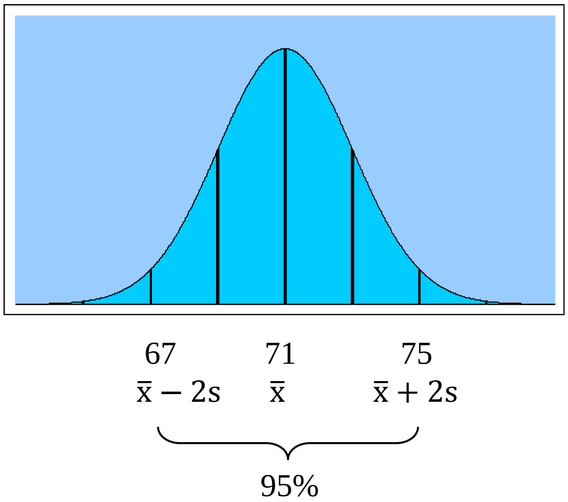
At this point we only know the empirical rule. As it turns out if we add and subtract one standard deviation (2 inches) to the mean we get
71 – 2 = 69 inches, which is the mean minus 1 standard deviation
71 + 2 = 73 inches, which is the mean plus 1 standard deviation



Since the heights of 69 inches to 73 inches are within 1 standard deviation of the mean and heights are normally distributed, 68% of adult males have heights between 69 and 73 inches.

b. What percentage of adult males are more than 75 inches tall?

71 + 2 + 2 = 75 inches 75 inches is the value 2 standard deviations above the mean.



The empirical rule tells us that 95% of the data is within 2 standard deviations of the mean. So, 100 – 95 = 5% of the data is outside of 2 standard deviations from the mean (more than 75 and less than 67). We only want the heights more than 75 inches, which is half of the 5% or 2.5%.

2.5% of adult males are taller than 75 inches.

Using the empirical rule is not only awkward, but we are limited to talking about values that are 1, 2, or 3 standard deviations from the mean. Also, different types of data yield different normal distributions because they have different means and different standard deviations. Next we will standardize normal distributions so that we can use one chart to determine percents of data and probabilities.

Z-scores:

The z-score measure the number of standard deviations between a data value and the mean.

z-score formula:

$$z = \frac{x - \bar{x}}{s}$$

x = data value we are interested in
$\bar{x}$ = mean
s = standard deviation
z = z-score

Examples:

2. For normally distributed data with a mean of 50 and standard deviation of 10, find the z-score for the following values:

a. value of 60

Steps	Reasons
$z = \frac{x - \bar{x}}{s}$	Write the formula for the normal distribution from the formula sheet.
$z = \frac{60 - 50}{10}$ $z = 1$	Plug in the data value, mean, and standard deviation. Calculate the z-score.

Because z is positive the value of 60 is 1 standard deviation more than the mean.

b. value of 32

Steps	Reasons
$z = \frac{x - \bar{x}}{s}$	Write the formula for the normal distribution from the formula sheet.
$z = \frac{32 - 50}{10}$ $z = -1.8$	Plug in the data value, mean, and standard deviation. Calculate the z-score.

Because z is negative the value of 32 is 1.8 standard deviations less than the mean.

Not only do z-scores tell us the number of standard deviations that a value is from the mean, it also lets us use one chart to calculate the percent of data less than a value, more than a value, or between two values when we know the data is normally distributed.

The normal distribution chart that comes with this course is what is known as a percentile chart. It will tell us the percent of data that is less than a given z-value.

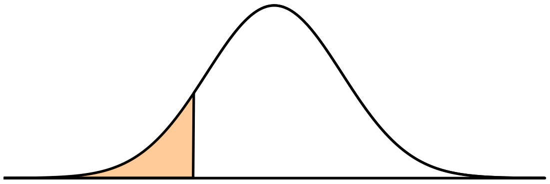
Examples:

3. To find the percent of data less than z = -2.47, go down the chart on the left to the z-score -2.4. Then go across to .07 to

Standard Normal Distribution Cumulative Probabilities (Percentiles)
Table Values represent area to the left of z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	**0.07**	0.08	0.09
-3.00	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	**0.0011**	0.0010	0.0010
-2.90	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	**0.0015**	0.0014	0.0014
-2.80	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	**0.0021**	0.0020	0.0019
-2.70	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	**0.0028**	0.0027	0.0026
-2.60	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	**0.0038**	0.0037	0.0036
-2.50	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	**0.0051**	0.0049	0.0048
-2.40	**0.0082**	**0.0080**	**0.0078**	**0.0075**	**0.0073**	**0.0071**	**0.0069**	**0.0068**	0.0066	0.0064

So, 0.0068 or 0.68% of the data is less than z=-2.47. That percentage is represented by the shaded region below. The chart always gives us the percent less or shaded area to the left of z.



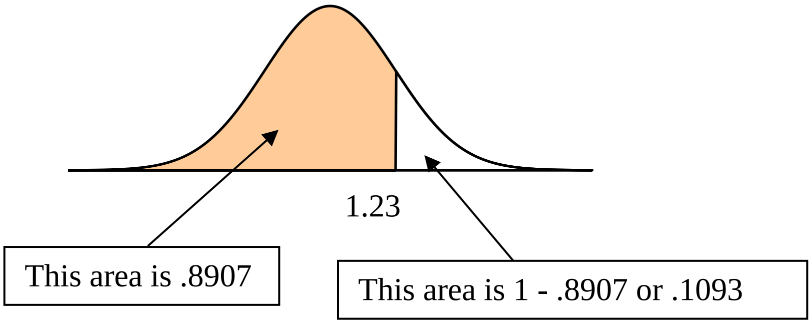
-2.47

4. To find the percent of values that is more than z = 1.23, we still use the chart:

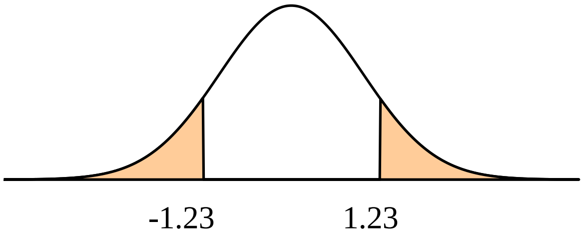
Standard Normal Distribution Cumulative Probabilities (Percentiles)
Table Values represent area to the left of z

Z	0.00	0.01	0.02	**0.03**	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	**0.5120**	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
1.00	0.8413	0.8438	0.8461	**0.8485**	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	**0.8708**	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	**0.8849**	**0.8869**	**0.8888**	**0.8907**	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

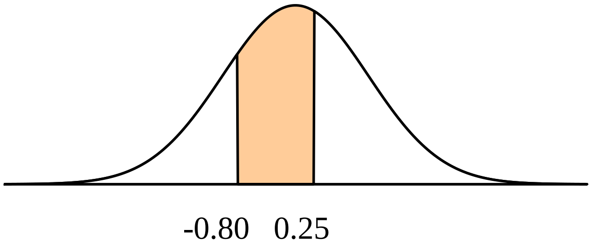
So, 0.8907 of the data is less than z = 1.23, but we are asked for the percent of data more than 1.23. Look at the picture of the normal distribution below



For the area more than a z-score, we just take 1 – value from the chart. So, .1093 or 10.93% of the data is more than a z-score of 1.23. We should also note that a score of more than 1.23 is the same as a z-score of less than 1.23. This is true because the normal distribution is symmetric as in the picture below.



5. Find the percent of values that is between z = -0.80 and z = 0.25.



Here we want to think of the area, which is also equal to the percent. The shaded area is all the area to left of z=0.25 minus the area to the left of z=-0.80. So, we will use the chart to look up both percentiles and find the difference.

Z	0.00	0.01	0.02	0.03	0.04	**0.05**	0.06	0.07	0.08	0.09
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	**0.5596**	0.5636	0.5675	0.5714	0.5753
0.20	**0.5793**	**0.5832**	**0.5871**	**0.5910**	**0.5948**	**0.5987**	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517

Z	**0.00**	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.90	**0.1841**	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.80	**0.2119**	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.70	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148

The percentile (or area) for z = .25 is 0.5987 and the percentile (or area) for z = -0.80 is 0.2119. We subtract the smaller area from the larger area to get the area between the two values. 0.5987 – 0.2119 = 0.3868

38.68% of the values are between z = -0.80 and z = 0.25

These last three examples really show us the three possibilities from the chart:

1. Percent less than a z-score: read it off the chart
2. Percent more than a z-score: 1 – values from the chart
3. Percent between two z-scores: look up both z-scores and find the difference

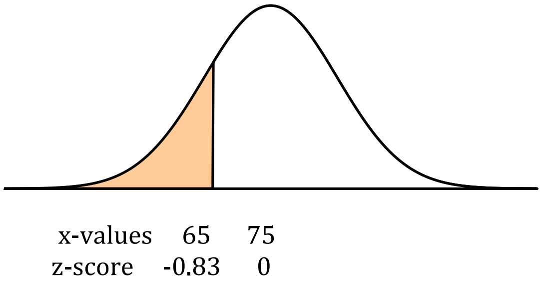
It is best to draw the bell curve when working with the normal distribution, but these three situations really do not change.

Since z-scores and the chart can be used to find the percent of data, they can also be used to find the probability of selecting a single value that is more than, less than, or between values when the data is normally distributed. The next step is to put everything together. We will take normally distributed values, calculate the z-scores, and use the chart to calculate percents or probabilities.

<u>Examples</u>:

For examples 6, 7, and 8, assume that life expectancy is normally distributed with a mean life expectancy of 75 years and a standard deviation of 12 years.

6. What percent of the population live less than 65 years?



Draw the bell-curve and find the z-score: $z = \frac{x-\bar{x}}{s} = \frac{65-75}{12} - 0.8333$ ...
Shade to the left for values less than. Use z = – 0.83 rounded off to the hundredth because the chart has values to the hundredth.

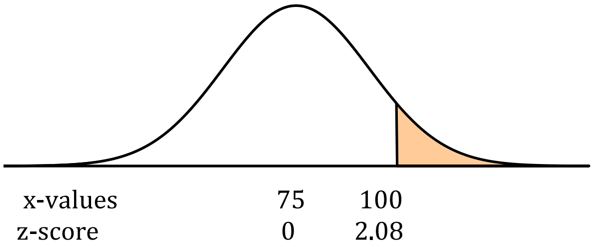
Z	0.00	0.01	0.02	**0.03**	0.04	0.05	0.06	0.07	0.08	0.09
-0.90	0.1841	0.1814	0.1788	**0.1762**	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.80	**0.2119**	**0.2090**	**0.2061**	**0.2033**	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.70	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148

The chart gives the percent of values less than. So, the answer is read from the chart.

0.2033 or 20.3% of the population live less than 65 years.

7. What is the probability that a randomly selected individual lives over 100 years?

First we need to note that the percent of people living over 100 years is the same as randomly selecting an individual that lives over 100 years. We continue as we have been with z-scores and the normal distribution chart.



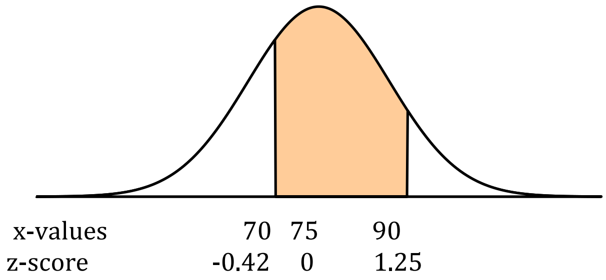
The chart always gives us area, percents, and probability for less than. To get the probability for more than 100 years we take 1 – the chart value.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	**0.08**	0.09
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	**0.9761**	0.9767
2.00	**0.9772**	**0.9778**	**0.9783**	**0.9788**	**0.9793**	**0.9798**	**0.9803**	**0.9808**	**0.9812**	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857

1 - 0.9812 = 0.0188

There is a 0.0188 or 1.88% probability that an individual will live more than 100 years.

8. What is the probability of selecting an individual that will live between 70 and 90 years?



When we use the normal distribution to find the percent or probability between two values, we find to z-scores, find two percentages, and take the difference. Looking at picture above, we see that the area of the shaded region is the area to the left of z=1.25 minus the unshaded area to the left of z=-0.42.

$\mathrm{z} = \frac{\mathrm{x}-\bar{\mathrm{x}}}{\mathrm{s}} = \frac{70-75}{12} = -0.4166 \ldots \text{use } \mathrm{z} = -0.42$ Chart is exact to the hundredth.

$\mathrm{z} = \frac{\mathrm{x}-\bar{\mathrm{x}}}{\mathrm{s}} = \frac{90-75}{12} = 1.25$

Z	0.00	0.01	0.02	0.03	0.04	**0.05**	0.06	0.07	0.08	0.09
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	**0.8749**	0.8770	0.8790	0.8810	0.8830
1.20	**0.8849**	**0.8869**	**0.8888**	**0.8907**	**0.8925**	**0.8944**	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

Z	0.00	0.01	**0.02**	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.50	0.3085	0.3050	**0.3015**	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.40	**0.3446**	**0.3409**	**0.3372**	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.30	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483

0.8944 – 0.3372 = 0.5572

There is a 0.5572 or 55.72% chance that an individual will live between 70 and 90 years.

Exercises

1. What does it mean that a height of 74 inches is the 95th percentile for men?

2. What are the characteristics of the normal distribution?

3. What does it mean that a height of 69 inches is the 92nd percentile for women?

4. What does it mean that a test score of 60 is the 85th percentile?

5. What does it mean that a test score of 52 is the 98th percentile?

Blood pressure is normally distributed. The units millimeter of mercury are represented by mm Hg. In a particular country the mean systolic blood pressure is 130 mm Hg with a standard deviation of 15 mm Hg. Use the empirical rule to answer the following questions about this particular country:

6. What percent of the population has a systolic blood pressure between 115 and 145 mm Hg?

7. What percent of the population has a systolic blood pressure between 100 and 160 mm Hg?

8. What percent of the population has a systolic blood pressure between 85 and 175 mm Hg?

9. What percent of the population has a systolic blood pressure more than 145 mm Hg?

10. What percent of the population has a systolic blood pressure less than 115 mm Hg?

11. What percent of the population has a systolic blood pressure less than 100 mm Hg?

12. What percent of the population has a systolic blood pressure more than 160 mm Hg?

Serum cholesterol levels are normally distributed. The mean serum cholesterol level for adult males is 210 mg/dL with a standard deviation of 45 mg/dL.

13. What percent of adult males have a serum cholesterol level between 165 mg/dL and 255 mg/dL?

14. What percent of adult males have a serum cholesterol level between 120 mg/dL and 300 mg/dL?

15. What percent of adult males have a serum cholesterol level less than 120 mg/dL?

16. What percent of adult males have a serum cholesterol level less than 165 mg/dL?

17. What percent of adult males have a serum cholesterol level more than 255 mg/dL?

18. What percent of adult males have a serum cholesterol level more than 255 mg/dL?

For normally distributed data with a mean of 72 and standard deviation of 6, find the z-score for the following data values. Round answers to the nearest hundredth.

19. 84

20. 66

21. 63

22. 87

23. 83

24. 68

25. 72

26. 58

27. 70

28. 91

For the standard normal distribution find the percent of z-scores for z:

29. less than z = 2.10

30. less than z = 1.57

31. less than z = -1.34

32. less than z = -2.31

33. more than z = 0.27

34. more than z = -0.35

35. more than z = -2.14

36. more than z = 0.59

37. more than z = -0.29

38. more than z = -3.04

39. between z = -1.36 and z = 2.08

40. between z = 0.17 and z = 1.23

41. between z = -0.09 and z = 3.01

42. between z = -2.12 and z = -1.08

The life span for light bulbs is normally distributed. The mean life span for a particular brand of light bulb is 975 hours with a standard deviation of 45 hours. For this particular brand of light bulbs, find the percent of light bulbs that have a life span of

43. less than 1047 hours

44. less than 1029 hours

45. more than 894 hours

46. more than 1074 hours

47. less than 890 hours

48. less than 920 hours

49. more than 1000 hours

50. more than 900 hours

51. between 975 and 1100 hours

52. between 925 and 975 hours

53. between 950 and 1000 hours

54. between 850 and 1050 hours

55. between 860 and 1035 hours

Assume that women’s heights are normally distributed with a mean of 64 inches and a standard deviation of 2.5 inches. Find the probability of randomly selecting a women who is

56. less than 69 inches tall.

57. more than 67 inches tall.

58. between 61 and 70 inches tall.

59. less than 68 inches tall.

60. more than 65.8 inches tall.

61. between 62.4 and 68.3 inches tall.

62. less than 64 inches tall.

63. more than 71.6 inches tall.

64. between 56.4 and 70.4 inches tall.

65. less than 56.4 inches tall.

66. more than 62.3 inches tall.

67. between 59.7 and 63.1 inches tall.

When we were graphing, we were studying the relationship between two variables. Those graphs could take on many shapes such as lines and parabolas. When we study the relationship between two variables, one of the first steps is to draw the relationship on the coordinate plane.

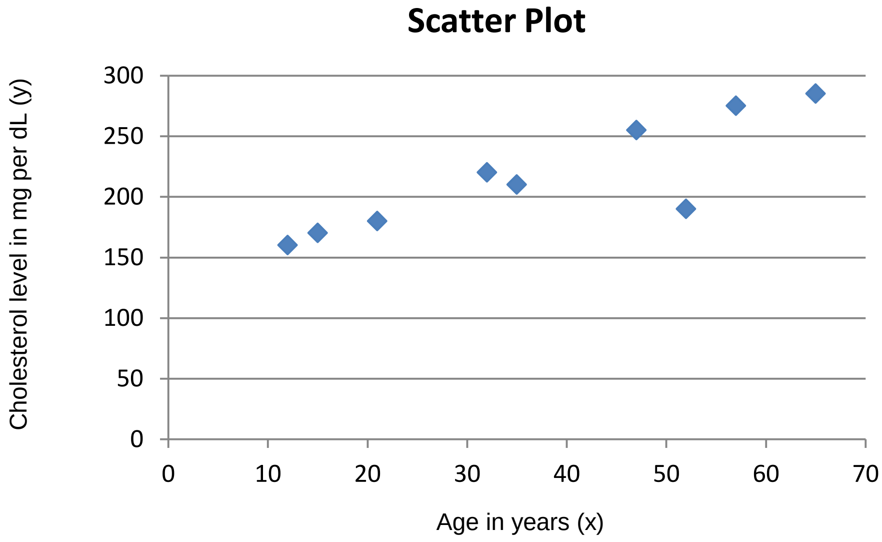
With a scatter plot we draw the points of two variables to see if there is a relation. The independent variable is represented by the x-coordinates along the horizontal axis. The dependent variable is represented by the y-coordinates along the vertical axis. The independent variable is used to predict the dependent variable. With correlation and regression, we do not show that the independent variable (x) causes the dependent variable (y) to act in a certain way. We are only showing whether or not the dependent variable can be used to predict the value of the independent variable.

Example:

1. Different people were studied to see if there is a relationship between age and cholesterol level. Cholesterol measures the milligrams (mg) of cholesterol per deciliter (dL) of blood. Use the data below to draw a scatter diagram to see if there is a relationship between age and cholesterol.

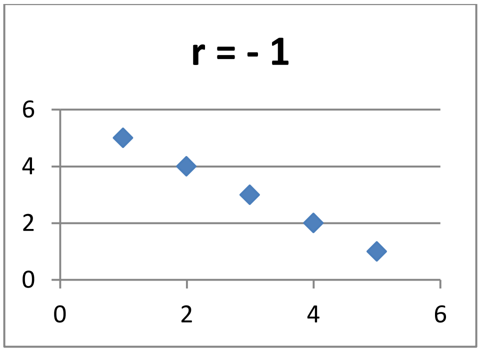
Age (x)	35	12	65	47	21	32	52	15	57
Cholesterol (y)	210	160	285	255	180	220	190	170	275

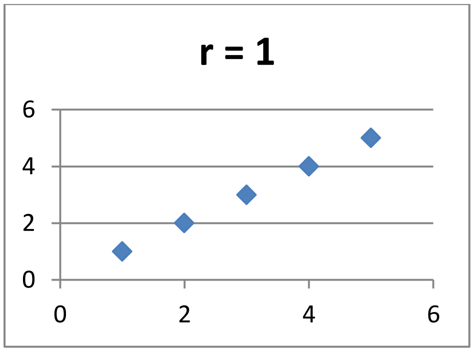
Graph the points (x,y) on a set of axes.

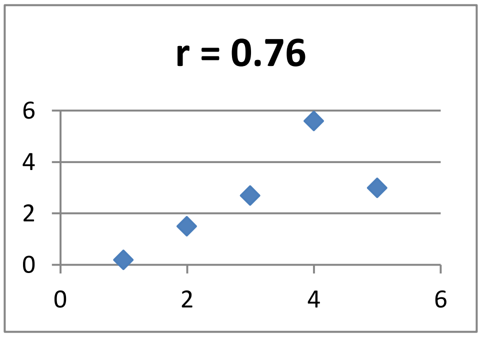


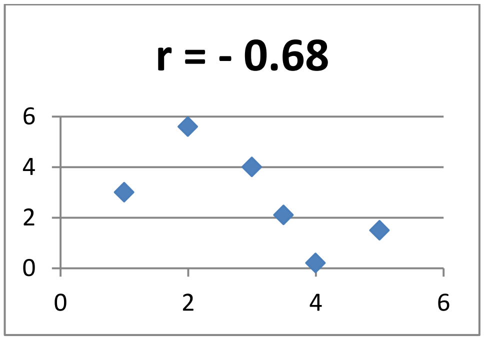
It is a good idea to label the axes with age and cholesterol level so that the reader knows what the variables are. Here we would be using the age (independent variable) to predict the cholesterol level (dependent variable). We are not saying that age is the cause or only cause of cholesterol level. It very well may be a factor, but there are clearly other factors such as heredity, diet, exercise, etc.

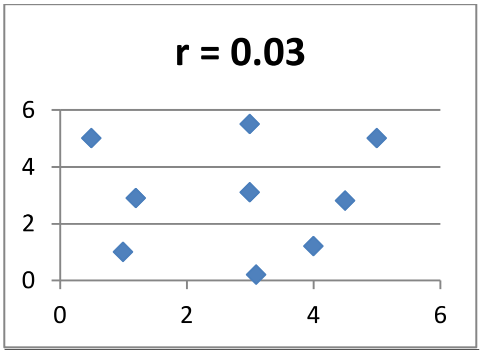
Once we have graphed the scatter plot, we can ask ourselves the question what is the shape of the graph. While not perfect the graph does look somewhat like a line, especially if we do not include the point (52,190). Linear correlation refers to the linear relationship between two variables. A strong linear correlation means that the scatter diagram points are close to a line. A weak linear correlation means that the scatter diagram points follow a line but not closely. The linear correlation coefficient or Pearson's coefficient (also called the correlation coefficient) ranges from r = -1 to 1. A value of r = 1 indicates there is perfect positive linear correlation, which is to say the scatter diagram points fall exactly on a line with positive slope ⤢. A linear correlation coefficient of r= -1 indicates that the scatter diagram points fall exactly on a line with a negative slope ⤡. If r = 0, there is no correlation.

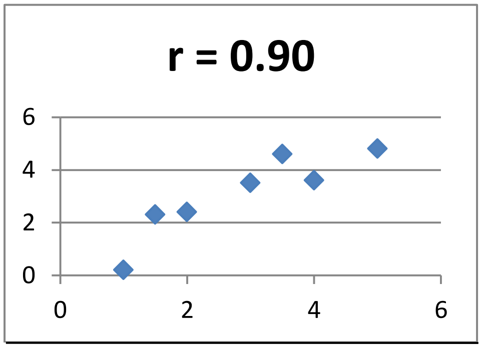

r = - 1


r = 1


r = 0.76


r = - 0.68


r = 0.03


r = 0.90

The formula for the linear correlation coefficient is the following:

$$r=\frac{n\left(\sum xy\right)-\left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\left(\sum x^2\right)-\left(\sum x\right)^2\right]\left[n\left(\sum y^2\right)-\left(\sum y\right)^2\right]}}$$

r is the linear correlation coefficient
n is the number of data values
x are the independent variables (x)
y are the dependent variables (y)

There is one example of how to find the correlation coefficient below, but it is much easier to use a scientific calculator to find the correlation coefficient.

Examples:

2. Using the same data from example 1, different people were studied to see if there is a relationship between age and cholesterol level. Cholesterol measures the milligrams (mg) of cholesterol per deciliter (dL) of blood. Use the data below to find the correlation coefficient.

Age (x)	35	12	65	47	21	32	52	15	57
Cholesterol (y)	210	160	285	255	180	220	190	170	275

$$r=\frac{n\left(\sum xy\right)-\left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\left(\sum x^2\right)-\left(\sum x\right)^2\right]\left[n\left(\sum y^2\right)-\left(\sum y\right)^2\right]}}$$

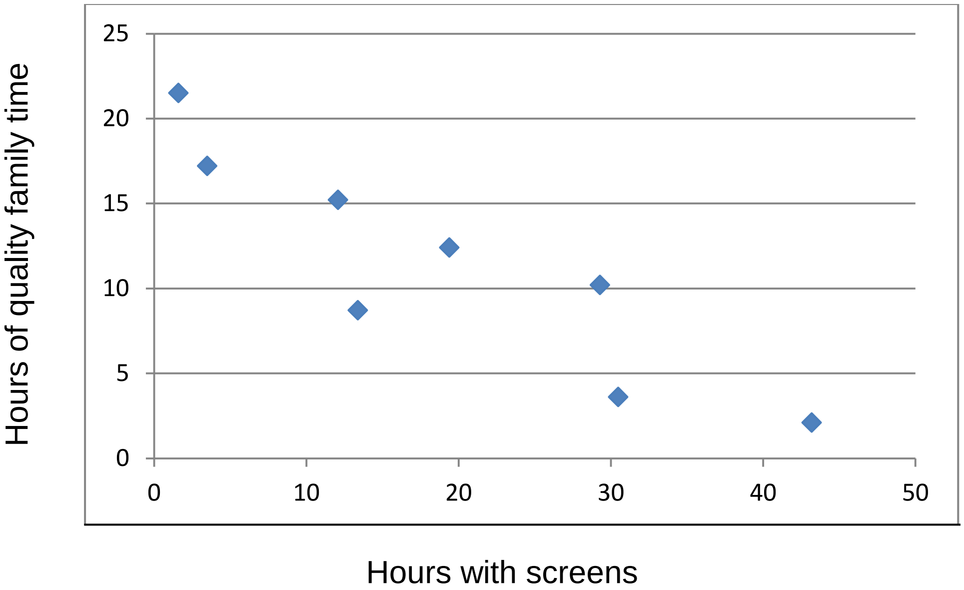
x	y	x^2	y^2	xy
35	210	1225	44100	7350
12	160	144	25600	1920
65	285	4225	81225	18525
47	255	2209	65025	11985
21	180	441	32400	3780
32	220	1024	48400	7040
52	190	2704	36100	9880
15	170	225	28900	2550
57	275	3249	75625	15675
336	1945	15446	437375	78705

$$r=\frac{9(78{,}705)-(336)(1945)}{\sqrt{\left[9(15{,}446)-(336)^2\right]\left[9(437{,}375)-(1945)^2\right]}}=\frac{54825}{63286.61233}=0.866297$$

$r=0.87$ is an indication that there is a strong positive correlation.

3. An experimenter believes that there is a relationship between the number of hours children spend with electronic devices with screens such as televisions, tablets, computers, or telephones and the amount of quality time spent with family. In the chart below the independent variable is the number of hours spent with screens and the dependent variable is the amount of quality time spent with family. Make a scatter plot, find the correlation coefficient, and comment on the correlation.

Hours with screens (x)	30.5	43.2	3.5	12.1	1.6	29.3	13.4	19.4
Hours of quality family time (y)	3.6	2.1	17.2	15.2	21.5	10.2	8.7	12.4



$$r = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\left(\sum x^2\right) - \left(\sum x\right)^2\right]\left[n\left(\sum y^2\right) - \left(\sum y\right)^2\right]}}$$

x	y	x^2	y^2	xy
30.5	3.6	930.25	12.96	109.8
43.2	2.1	1866.24	4.41	90.72
3.5	17.2	12.25	295.84	60.2
12.1	15.2	146.41	231.04	183.92
1.6	21.5	2.56	462.25	34.4
29.3	10.2	858.49	104.04	298.86
13.4	8.7	179.56	75.69	116.58
19.4	12.4	376.36	153.76	240.56
153	90.9	4372.12	1339.99	1135.04

$$r = \frac{8(1135.04) - (153)(90.9)}{\sqrt{\left[8(4372.12) - (153)^2\right]\left[8(1339.99) - (90.9)^2\right]}}$$

$$r = \frac{-4827.38}{5331.3929}$$

$$r = -0.90546$$

Both the scatter plot and the correlation coefficient of r = – 0.905 indicate that there is a strong negative correlation. Remember it is much easier to calculate the correlation coefficient using a scientific calculator.

Exercises

For the following sets of data, draw a scatter diagram. From the scatter diagram decide whether there is positive linear correlation, negative linear correlation, or no linear correlation.

1. A city wants to see if there is a linear relationship between property tax and annual income of its residents. A random sample of seven residents was selected with the following results:

Property Tax in thousands of dollars (X)	3.5	5.2	9.4	7.1	1.2	15.8	6.3
Annual Income thousands of dollars (Y)	55	65	125	102	23	205	85

2. A county relief organization wants to determine if there is a linear relationship between the number of children and annual household income in thousands of dollars.

Number of children (X)	3	6	2	5	1	2	4
Annual Income thousands of dollars (Y)	47	21	75	30	94	81	35

3. A fraternity at a large university wanted to determine if there is a linear relationship between the number of alcoholic beverages consumed per week not including weekends and the grade point average (GPA) of its members.

Number of Alcoholic Beverages Consumed (X)	4	18	7	0	10	6	12	3	15
Grade Point Average (Y)	3.8	2.3	3.5	3.7	2.8	4.0	3.2	3.6	2.9

4. A psychologist wants to find out if there is a linear relationship between the number of sunny days in a year and positive attitude. The psychologist measures positive attitude on a 100 point scale where a score of 100 indicates the highest possible positive attitude and 0 indicates a completely negative attitude (or the lowest possible positive attitude).

Number of Sunny Days (X)	240	315	142	85	330	220	123	98	57
Psychologists Attitude Test (Y)	73	89	65	25	68	78	81	58	32

5. A high school soccer coach wants to determine if there is a linear relationship between the number of days of practice and the number of wins in a season. She compares wins and days of practice for her last six seasons.

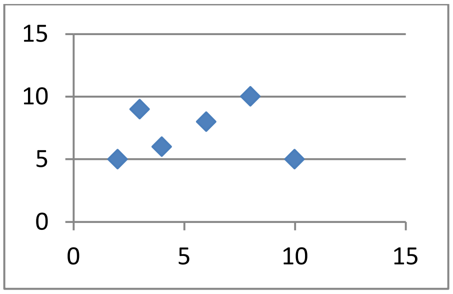
Number of Days of Practice (X)	45	56	48	60	52	58
Number of Wins (Y)	5	8	9	5	7	8

6. A regional manager for McDonald's believes that there is a linear relationship between a town's population and the number of successful McDonald's franchises in a town.

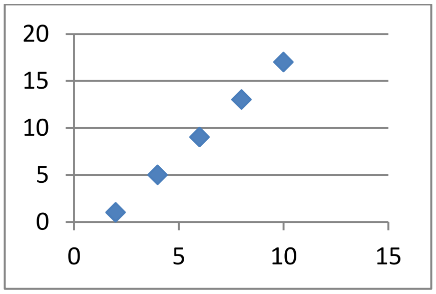
Population in tens of thousands (X)	25	37	6	15	11	9	53	3	28
Number of successful franchises (Y)	12	17	3	14	7	3	21	4	17

For the following problems, consider the following scatter plots:

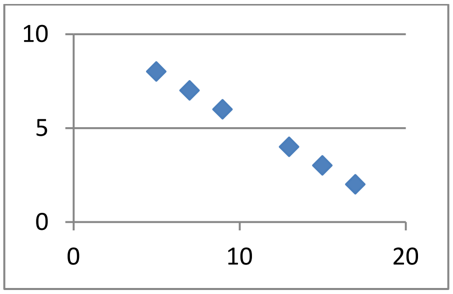
A



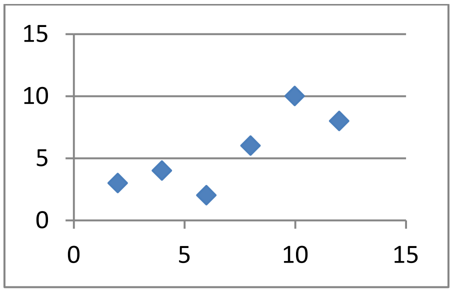
B



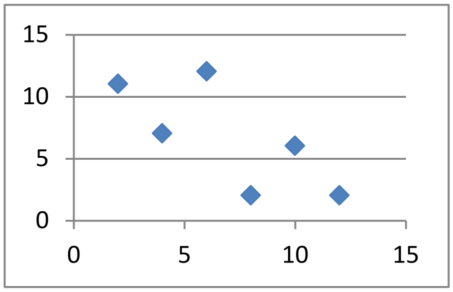
C



D



E



State which graph has the following correlation coefficient.

7. r = 0.82

8. r = -1.00

9. r = - 0.72

10. r = 1.00

11. r = 0.08

For the following problems you may use a calculator that finds the correlation coefficient or the formula. It is easier and quicker to use a calculator with the correlation coefficient feature.

12. Use the data from problem number 1 to find the correlation coefficient. According to the correlation coefficient is the correlation strong, moderate, or weak? Does the correlation coefficient support what you said about whether there is positive linear correlation, negative linear correlation, or no linear correlation?

13. Use the data from problem number 2 to find the correlation coefficient. According to the correlation coefficient is the correlation strong, moderate, or weak? Does the correlation coefficient support what you said about whether there is positive linear correlation, negative linear correlation, or no linear correlation?

14. Use the data from problem number 3 to find the correlation coefficient. According to the correlation coefficient is the correlation strong, moderate, or weak? Does the correlation coefficient support what you said about whether there is positive linear correlation, negative linear correlation, or no linear correlation?

15. Use the data from problem number 4 to find the correlation coefficient. According to the correlation coefficient is the correlation strong, moderate, or weak? Does the correlation coefficient support what you said about whether there is positive linear correlation, negative linear correlation, or no linear correlation?

16. Use the data from problem number 5 to find the correlation coefficient. According to the correlation coefficient is the correlation strong, moderate, or weak? Does the correlation coefficient support what you said about whether there is positive linear correlation, negative linear correlation, or no linear correlation?

17. Use the data from problem number 6 to find the correlation coefficient. According to the correlation coefficient is the correlation strong, moderate, or weak? Does the correlation coefficient support what you said about whether there is positive linear correlation, negative linear correlation, or no linear correlation?

Exercise Set 1.1

1. $7 < 9$

3. $-8 > -12$

5. $-4 < -3$

7. $12 < |-15|$

9. $|-6| > |4|$

11. $-1, |0|, |-2|, 3, 7, |-11|$

13. $-5, -2, 0, 1, |-6|, |-7|, |11|$

15. -2

17. -21

19. 4

21. -42

23. -7

25. -9

27. -12

29. -5

31. -45

33. -40

35. -63

37. -9

39. -9

41. -7

43. -20

45. 125

47. 16

49. 64

51. -16

53. 17

55. -3

57. -3

59. -28

61. 6

63. -11

65. -9

67. 48

69. -75

71. -304

73. -395 Maria borrows $395.

75. 2528 years

Exercise Set 1.2

1. $2^2 \cdot 3^2$

3. $2^2 \cdot 5^2$

5. $2^2 \cdot 5 \cdot 11$

7. 90

9. 1080

11. 7560

13. 6

15. 56

17. 84

19. 630 students

21. 24 exams

Exercise Set 1.3

1. $\frac{7}{2}$

3. $\frac{53}{8}$

5. $-\frac{39}{7}$

7. $-\frac{107}{7}$

9. $4\frac{3}{5}$

11. $8\frac{5}{9}$

13. $-4\frac{3}{4}$

15. 0.625

17. $9.\overline{3}$

19. 3.2

21. -12.1875
23. $-16.\overline{5}$

25. $32.8\overline{3}$

27. $\frac{57}{100}$

29. $\frac{532}{100}$

31. $\frac{562}{1000}$

33. $\frac{3}{7}$

35. $\frac{5}{6}$

37. $\frac{1}{2}$

39. $\frac{4}{15}$

41. $\frac{1}{9}$

43. $-\frac{4}{21}$

45. $\frac{2}{15}$

47. $12\frac{1}{4}$

49. $-39\frac{5}{27}$

51. $27\frac{1}{2}$

53. $\frac{5}{6}$

55. $-1\frac{1}{2}$

57. $\frac{128}{375}$

59. $-1\frac{1}{3}$

61. $1\frac{11}{16}$

63. $-1\frac{7}{8}$

65. $\frac{1}{3}$

67. $\frac{4}{9}$

69. $\frac{25}{49}$

71. $7\frac{9}{16}$

73. $\frac{5}{7}$

75. $\frac{5}{9}$

77. $\frac{11}{12}$

79. $\frac{19}{60}$

81. $-\frac{8}{15}$

83. $-\frac{67}{75}$

85. $14\frac{1}{2}$

87. $-4\frac{11}{18}$

89. $8\frac{7}{12}$

91. $-2\frac{1}{18}$

93. $-\frac{1}{18}$

95. $-\frac{5}{23}$

97. $\frac{1}{3}$

99. $\frac{37}{60}$

101. $-\frac{7}{20}$

103. $\frac{5}{18}$

105. $1\frac{7}{15}$

107. $-6\frac{11}{14}$

109. $7\frac{3}{5}$ gallons of gas

Exercise Set 1.4

1. 7

3. 8

5. 36

7. 12.530

9. $4{,}488.943$

11. $4\sqrt{2}$

13. $8\sqrt{2}$

15. $15\sqrt{3}$

17. $165\sqrt{2}$

19. $3\sqrt{10}$

21. $10\sqrt{21}$

23. $196\sqrt{3}$

25. $420\sqrt{6}$

27. $5\sqrt{3}+5$

29. $7\sqrt{2}-7$

31. $30+45\sqrt{3}$

33. $70\sqrt{3}-56$

35. $9\sqrt{3}$

37. $9\sqrt{5}-7\sqrt{10}$

39. $13\sqrt{3}$

41. $8\sqrt{2}$

43. a. 792.96 km/hour
    b. 340.05 km/hour
    c. As the depth of the water decreases, the velocity of the Tsunami decreases.

45. 8.82 seconds

Exercise Set 1.5

1. 81

3. 16

5. -16

7. 1

9. -1

11. 64

13. 32

15. 81

17. 4096

19. $\frac{1}{16}$

21. $-\frac{1}{25}$

23. $\frac{1}{16}$

25. -1

27. $\frac{1}{27}$

29. 64

31. $\frac{1}{8}$

33. $\frac{1}{27}$

35. $\frac{1}{49}$

37. y^7

39. $\frac{1}{t^5}$

41. $\frac{1}{y^8}$

43. x^5y^4

45. $\frac{x^2}{9}$

47. x^4y^4

49. $\frac{81}{y^4}$

51. $\frac{1}{xy^8}$

53. $\frac{y^5}{x^6}$

55. $3.74 \text{ x } 10^{12}$

57. $7.624 \text{ x } 10^{-9}$

59. $1.23 \text{ x } 10^{11}$

61. $6.2 \text{ x } 10^{-6}$

63. $5.85 \text{ x } 10^{20}$

65. $2.6784 \text{ x } 10^{18}$

67. $3.1 \text{ x } 10^{-16}$

69. $2.5 \text{ x } 10^{-11}$

71. $2 \text{ x } 10^0$ or 2

73. $9.11 \text{ x } 10^{72}$ grams or $9.11 \text{ x } 10^{69}$ kilograms

75. $1.47 \text{ x } 10^{19}$ miles

Exercise Set 2.1

1. 7

3. -5

5. 57

7. $-2\frac{41}{100}$

9. -84

11. 2

13. $\frac{2}{9}$

15. $-\frac{2}{3}$

17. -45

19. 90

21. -4

23. 75

25. -18

27. a. 71.3 inches
b. The formula overestimates the actual height by 0.3 inches. This trend is not likely to continue indefinitely or men's average height would continually increases. For instance the formula indicates that the average height of men will be about eighteen and a half feet 5000 years from now.

29. a. 18,000 students
b. The formula overestimates the actual number of students by 200.

Exercise Set 2.2

1. -10xy

3. xy

5. $12 + 3\mathrm{x}$

7. $-10\mathrm{y} - 10$

9. $-1.55\mathrm{t} - 8.62$

11. $5\mathrm{x} - 7\mathrm{y} - 2$

13. $6.34\mathrm{x} + 2.5\mathrm{y} - 18.11$

15. $20\mathrm{x} - 8\mathrm{y} + 12$

17. $-20\mathrm{x} + 15\mathrm{y} + 40$

19. $-2\mathrm{x} + 7\mathrm{y} - 8$

21. $3\mathrm{x} - 15\mathrm{y}$

23. $-8s - 14\mathrm{t}$

25. $-8\mathrm{x}^2 + 11x$

27. $9x^2 - 12x$

29. $-3x^2y + 7xy^2$

31. $-3x + 29$

33. $13x + 1$

35. $5x + 11$

37. $x - 4$

39. $11x - 29$

41. $4x^2 + 2$

43. $2x^2 - 6$

45. $4x^2 + 2x - 12$

47. $4x^2 + 2x$

49. $7x^2 - 18$

51. $-12x - 28$

53. $-20x + 64$

55. $10x + 119$

57. $4x - 47$

59. $y^2 - 9y + 21$

Exercise Set 2.3

1. $x = 10$

3. $x = -14$

5. $x = 1\frac{5}{12}$

7. $x = -0.12$

9. $x = -\frac{7}{8}$

11. $x = -7$

13. $x = 4$

15. $x = -3.5$

17. $x = -\frac{6}{7}$

19. $x = 4$

21. $x = -4$

23. $x = 2$

25. $x = -5$

27. $x = -16$

29. $x = -4$

31. $x = -1$

33. $x = -8$

35. $x = -1$

37. $x = 2\frac{1}{5}$

39. $x = 1\frac{5}{7}$

Exercise Set 2.4

1. $L = \frac{A}{W}$

3. $L = \frac{P-2W}{2}$.

5. $x = \frac{24+4y}{3}$

7. $y = \frac{28-4x}{-7}$.

9. $y = 3A - x - z$.

11. $r = \frac{t-Ms}{M}$

13. $C = \frac{5}{9}(F - 32)$.

15. $t = \frac{A-P}{Pr}$

Exercise Set 3.1

1. $5 + 3x$

3. $2x - 5$

5. $-5x - 5$

7. $-1.88x - 5$

9. $50 - z$

11. $40 - 2x$

13. 48

15. $1\frac{5}{12}$

17. 14 and 26

19. -32 and 25

Exercise Set 3.2

1. The odd integers are 15, 17, and 19.

3. The integers are 18, 19, and 20.

5. The stamp collector purchased thirty 5¢ stamps, ninety 15¢ stamps, and forty 25¢ stamps.

7. In 1968 the average salary for a college instructor was $10,040.

9. 35 cartridges were used over the life of the printer.

11. It will take 57 hours of training for somebody with a SAT Math score of 529 to raise their SAT Math score to 700 if we accept the SAT training program’s claim.

13. After sixty hours the costs of the two plans will be the same.

15. After 280 kilowatt hours the costs of the two plans will be the same.

Exercise Set 3.3

1. 0.8 EUR : 1 USD

3. 1.24 CAD : 1 USD

5. $x = 16$

7. $x = 30$

9. $x = 12$

11. $x = 23$

13. 88.2 miles

15. There are about 245 elephants.

17. There are about 63 bears in the wildlife refuge.

19. After 19.5 more months, the employee will have accrued enough vacation days (25 days) to take a five week vacation.

21. An additional $2250 must be invested so that $450 is earned each year.

Exercise Set 3.4

1. 0.12

3. 0.0035

5. $\frac{9}{20}$

7. $\frac{143}{300}$

9. 28%

11. $55\frac{5}{9}\%$

13. $333\frac{1}{3}\%$

15. 73%

17. 16.82%

19. 1229%

21. 43.2

23. 18

25. 52%

27. 192

29. 75

31. 64%

33. There was an 18% increase.

35. There was a 15% decrease.

37. The sale price of the blender is $48.75

39. The regular price of the coffee maker is $132.

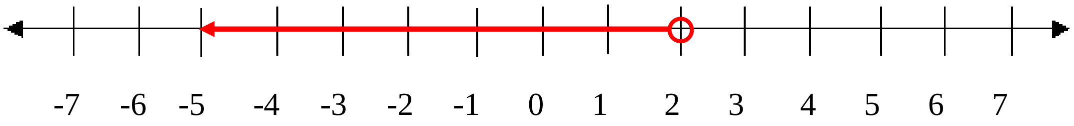
41. The value of the car after the first year is $25,004.

43. The wine shop’s sale price for the French Chardonnay is $11.70.

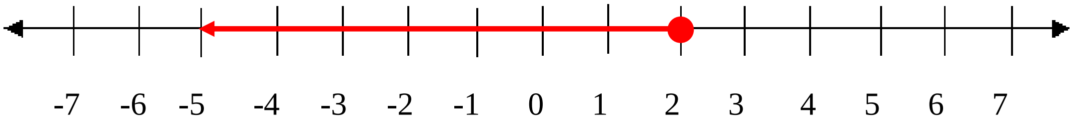
45. The supermarket’s mark-up rate is 65%.

Exercise Set 3.5

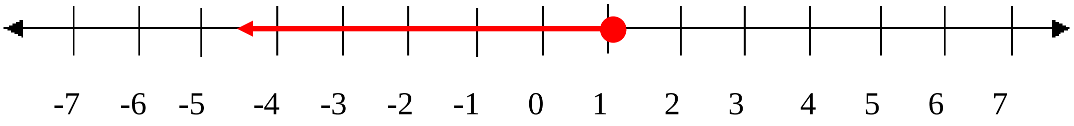
1. $x < 2$



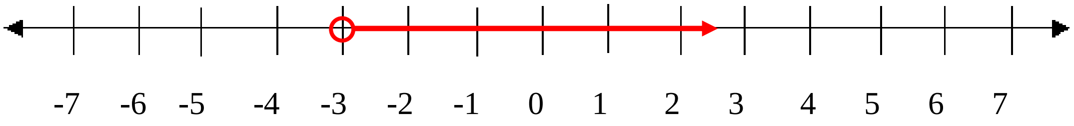
3. $x \leq 2$



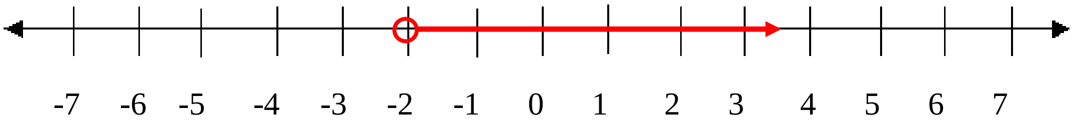
5. $x \leq 1$



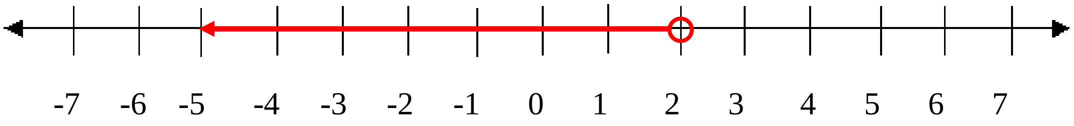
7. $x > -3$



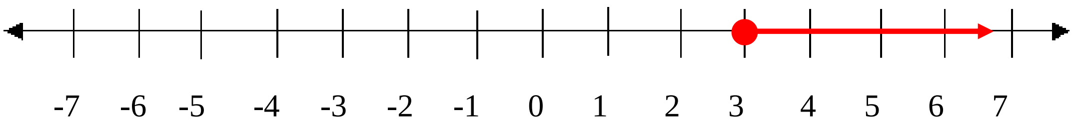
9. $x > -2$



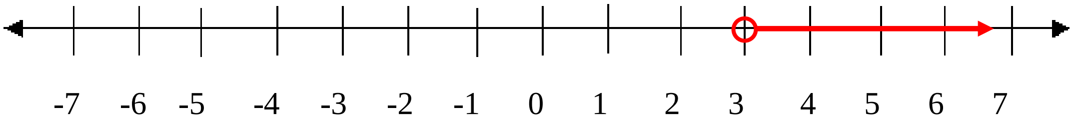
11. $x < 2$



13. $x \geq 3$



15. $x > 3$



17. $\{x|x < -7.5\}$

19. $\{x|x > 3\}$

21. $\{x|x \leq -15\}$

23. $\left\{x \middle| x \leq -\frac{4}{3}\right\}$

25. $\{x|-1 < x \leq 3\}$

27. $\left\{x \middle| -\frac{1}{5} < x < \frac{9}{5}\right\}$

29. The student needs at least an 89 on the fifth test to earn an A in the math class.

31. At most 640 employees can be trained.

33. At most $9\frac{7}{9}$ ounces of artificial flavors can be added to the real orange juice.

35. $68° \leq F \leq 95°$

Exercise Set 4.1

1. $10x^3 + 15x^2 - 20x$

3. $21x^3 - 35x^2 - 77x$

5. $-6x^4 + 9x^3 + 18x^2$

7. $-8x^4 + 20x^3 + 24x^2$

9. $15x^8 - 9x^7 - 6x^6$

11. $12x^3y - 8x^2y^2 + 20xy^3$

13. $-45x^3y^3 + 30x^4y^2 + 24x^4y^3$

15. $40x^4y^2 - 80x^3y^2 - 56x^3y^3 - 96x^4y^3$

17. $15x^2 + 22x + 8$

19. $15x^2 + 29x - 14$

21. $35x^2 - 87x + 22$

23. $42x^2 - 53x + 15$

25. $55x^2 - 74x + 24$

27. $72x^2 - 15x - 42$

29. $15x^2 + 30x - 120$

31. $6x^4 - 6x^2 - 36$

33. $72x^4 - 206x^2 + 140$

Exercise Set 4.2

1. $6x(2x^2 + x - 3)$ or $6x(2x + 3)(x - 1)$

3. $4x^3(5x^2 - 4x - 3)$

5. $11x^2(3x^4 + 5x^2 - 4)$

7. $9x^2(7x^3 - 2x^2 + 6)$

9. $y^2(40y^4 - 49y^2 - 35)$

11. $(x + 2)(x + 3)$

13. $(x + 8)(x + 1)$

15. $(x - 4)(x + 2)$

17. does not factor

19. $(x - 3)(x - 4)$

21. $(x - 5)(x + 4)$

23. $(x + 7)(x - 3)$

25. $(x - 4)(x - 1)$

27. $(x + 9)(x + 2)$

29. $(x - 8)(x + 3)$

31. $(x + 6)(x - 5)$

33. does not factor

35. $(x + 8)(x + 8)$

37. $(2x - 1)(x + 3)$

39. $(3x - 1)(x - 5)$

41. does not factor

43. $(2x + 3)(x - 1)$

45. $(5x + 2)(x - 3)$

47. $(3x - 1)(2x + 1)$

49. $(4x + 3)(x - 2)$

51. $(2x - 5)(2x + 3)$

53. $5x(x + 3)(x + 1)$

55. $3x^2(x - 5)(x + 2)$

57. $4x^3(x - 4)(x - 2)$

59. $20x^3(x + 4)(x + 1)$

61. $x = 7,9$

63. $x = -\frac{3}{2}, \frac{1}{3}$

65. $x = 1,4$

67. $x = -7,3$

69. $x = -1,6$

71. $x = -3,4$

73. $x = 3,6$

75. $x = 0,1,2$

77. $x = -4,0,5$

79. $x = -\frac{1}{3}, 2$

81. $x = -\frac{3}{2}, 1$

83. The numbers are 5 and 6.

85. The numbers are 3 and 5.

Exercise Set 4.3

1. $x = -1, \frac{5}{3}$

3. $x = \frac{-3 \pm \sqrt{29}}{10}$

5. $x = \frac{7 \pm \sqrt{13}}{6}$

7. $x = \frac{3 \pm \sqrt{5}}{2}$

9. $x = \frac{1 \pm \sqrt{85}}{6}$

11. $x = \frac{1 \pm 2\sqrt{7}}{3}$

13. $x = -\frac{1}{6}, \frac{1}{2}$

15. $x = \frac{-1 \pm \sqrt{5}}{4}$

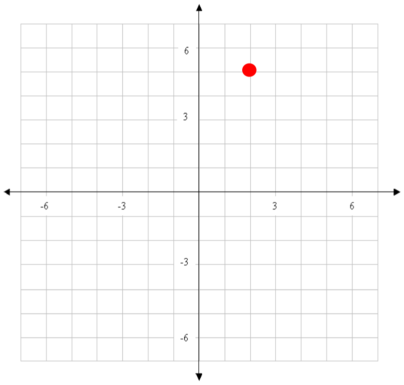
17. $x = -\frac{2}{3}, 2$

19. $x = \frac{7 \pm \sqrt{37}}{4}$

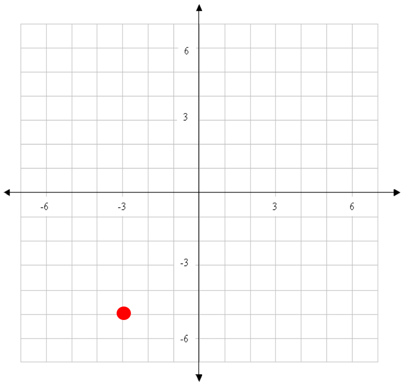
21. $x = 1 \pm \sqrt{6}$

Exercise Set 5.1

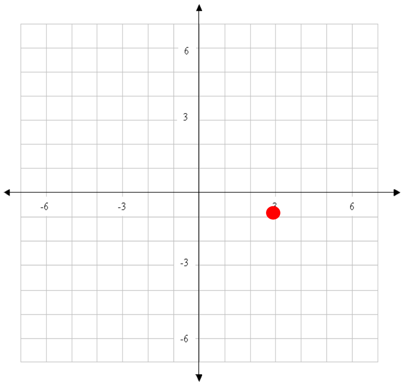
1. $(2,5)$



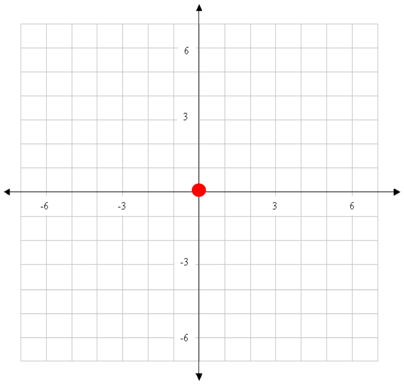
3. $(-3,-5)$



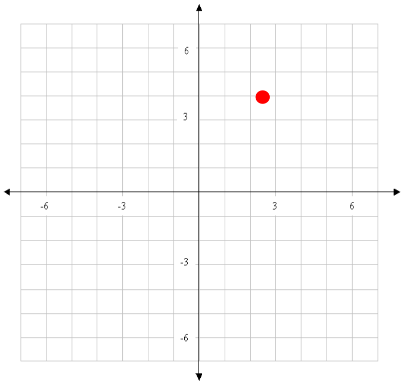
5. $(3,-1)$



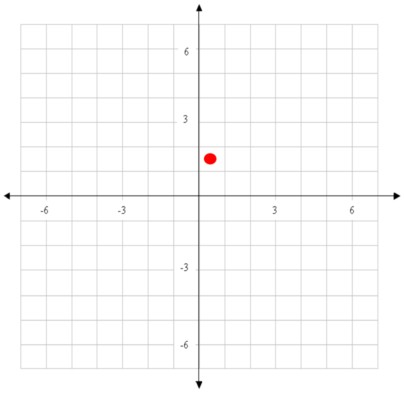
7. $(0,0)$



9. $(2.5,3.5)$



11. $\left(\frac{1}{2},\frac{3}{4}\right)$



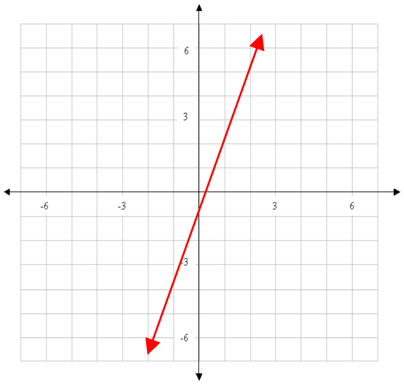
13.
a. $f(2) = 1$
b. $f(10) = 25$
c. $f(-3) = -14$
d. $f(0) = -5$

15.
a. $g(2) = 14$
b. $g(-3) = -6$
c. $g(0) = 0$
d. $g\left(\frac{1}{5}\right) = 1\frac{1}{25}$

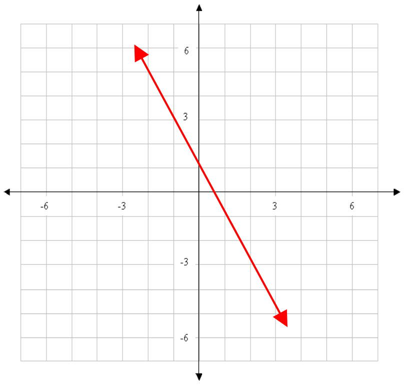
17.
a. $f(3) = 35$
b. $f(-2) = 20$
c. $f(0) = 8$
d. $f(-5) = 83$

19. a. $h(7) = 4$
b. $h(-5) = 2$
c. $h(0) = 3$
d. $h(91) = 10$

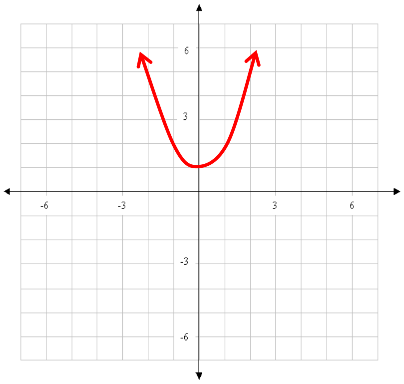
21. $f(x) = 3x - 1$



23. $f(x) = -2x + 1$



25. $g(x) = x^2 + 1$



27. Yes, the graph is the graph of a function.

29. Yes, the graph is the graph of a function.

31. No, the graph is not the graph of a function.

33. No, the graph is not the graph of a function.

35. Yes, the graph is the graph of a function.

37. a. 7725 kilograms
    b. 193,125 kilograms

39. a. $48,550
    b. The function underestimates the price by $3450.

Exercise Set 5.2

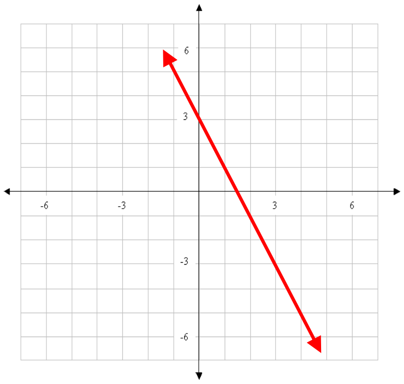
1. Yes, $(3,5)$ is a solution to $y = 2x - 1$.

3. No, $(5, -4)$ is not a solution to $y = -2x + 3$.

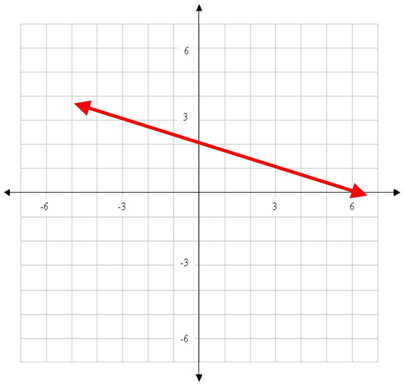
5. Yes, $(-2,3)$ is a solution to $y = 2x + 7$.

7. No, $(-2,5)$ is not a solution to $3x + 2y = 9$.

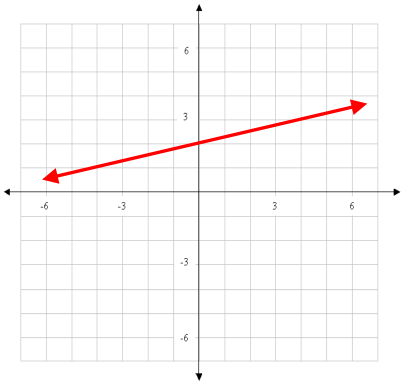
9. $y = -2x + 3$



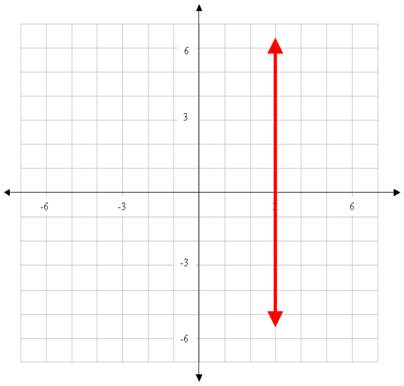
11. $y = -\frac{1}{3}x + 2$



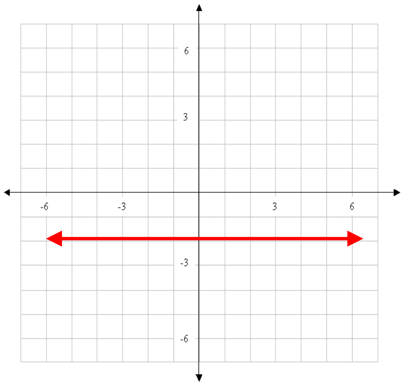
13. $y = \frac{1}{4}x + 2$



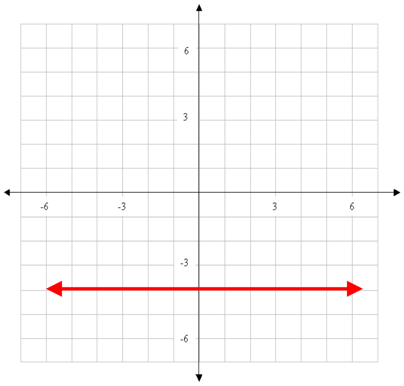
15. $x = 3$



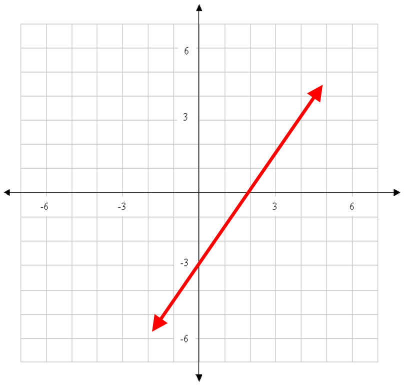
17. $y = -2$



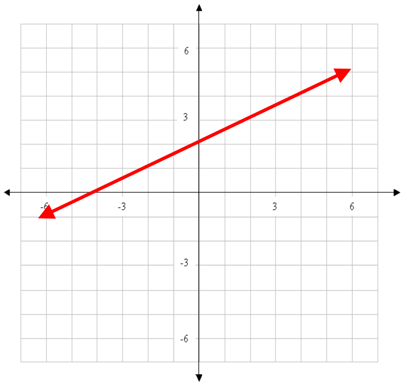
19. $y = -4$



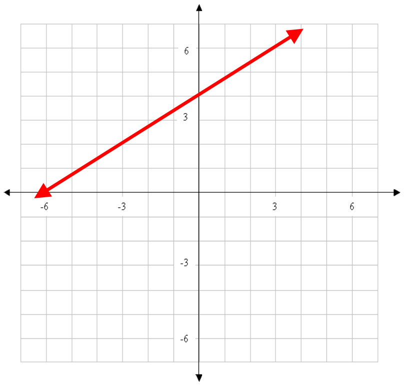
21. $3x - 2y = 6$



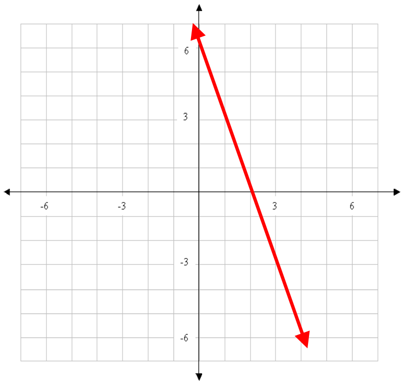
23. $2x - 4y = -8$



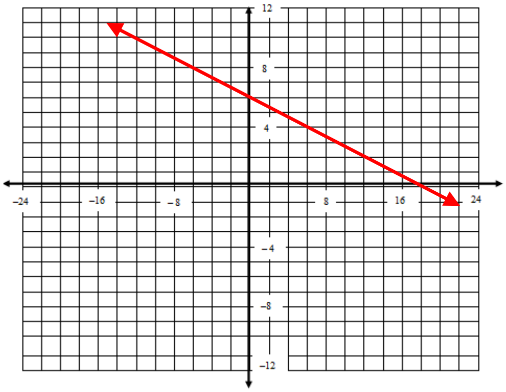
25. $-2x + 3y = 12$



27. $y = -3x + 6$



29. $y = -\frac{1}{3}x + 6$



31. $m = 4$

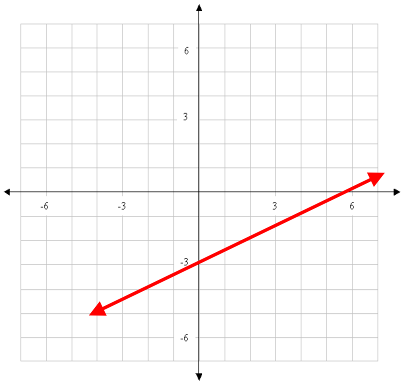
33. $m = -3$

35. $m = -\frac{1}{2}$

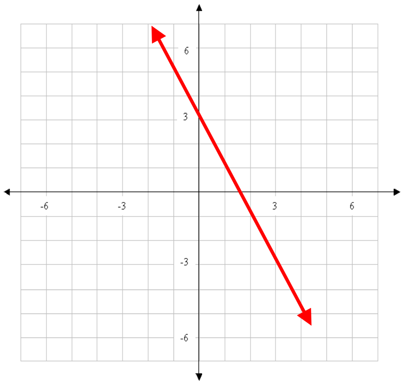
37. $m = 0$

39. undefined slope

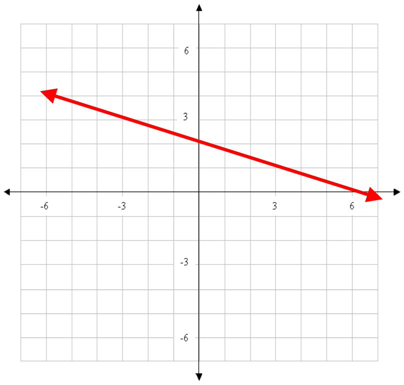
41. $y = \frac{1}{2}x - 3$



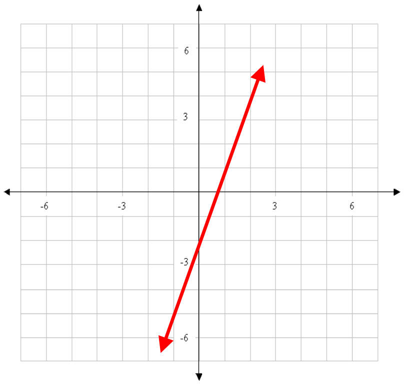
43. $y = -2x + 3$



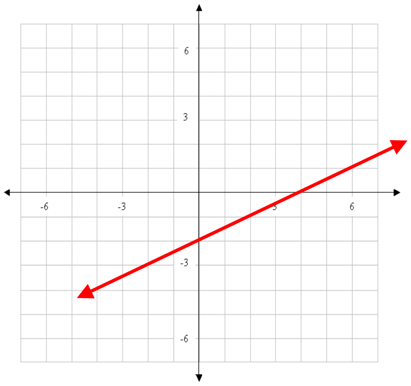
45. $y = -\frac{1}{3}x + 2$



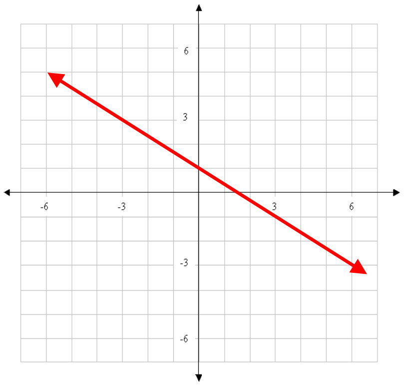
47. $y = 3x - 2$



49. $3x - 6y = 12$



51. $2x + 3y = 3$



Exercise Set 5.3

1. Yes

3. No

5. $(3,1)$

7. No solution

9. $(1,5)$

11. $(-1,1)$

13. $(2,3)$

15. $(-2,6)$

17. $(-5,-2)$

19. $(3,1)$

21. $(-3,2)$

23. $(1,4)$

25. $(2,-1)$

27. $(-1,-2)$

29. $(7,-3)$

31. $(-3,2)$

33. There are 52 grams of sugar in the cola and 45 grams of sugar in the root beer.

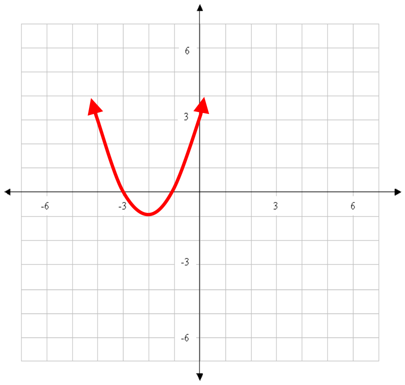
35. 21 motorcycles and 17 cars had all tires replaced.

37. The 4% bond is for $120,000 and the 2% bond is for $30,000.

39. The hot dog costs 1.75 euro and the beer costs 1.50 euro.

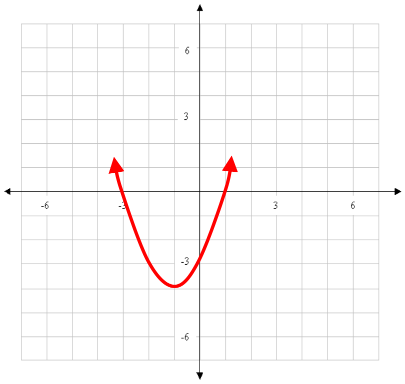
Exercise Set 5.4

1. $f(x) = x^2 + 4x + 3$
a. up
b. $(-2,-1)$
c. $(-3,0)$ and $(-1,0)$
d. $(0,3)$
e.



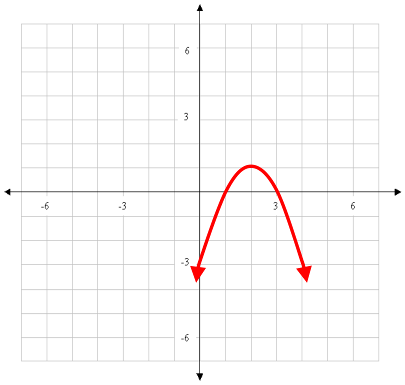
3. $f(x) = x^2 + 2x - 3$

a. up
b. $(-1, -4)$
c. $(-3,0)$ and $(1,0)$
d. $(0, -3)$
e.

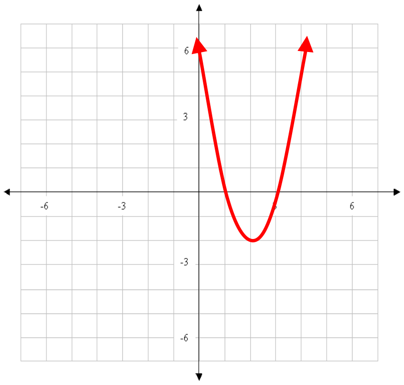


5. $f(x) = -x^2 + 4x - 3$

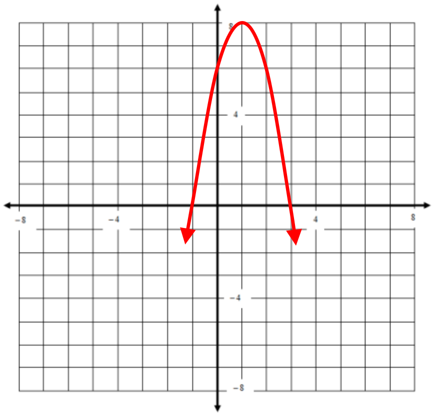
a. down
b. $(2,1)$
c. $(1,0)$ and $(3,0)$
d. $(0, -3)$
e.



7. $f(x) = 2x^2 - 8x + 6$
a. up
b. $(2, -2)$
c. $(1,0)$ and $(3,0)$
d. $(0,6)$
e.



9. $f(x) = -2x^2 + 4x + 6$
a. down
b. $(1,8)$
c. $(-1,0)$ and $(3,0)$
d. $(0,6)$
e.



11. The minimum of -34 is found at x = - 3.

13. The maximum of 42 is found at x = - 2.

15. The minimum of -140 is found at x = 4.

17. The maximum of 82,500 is found at x = 25.

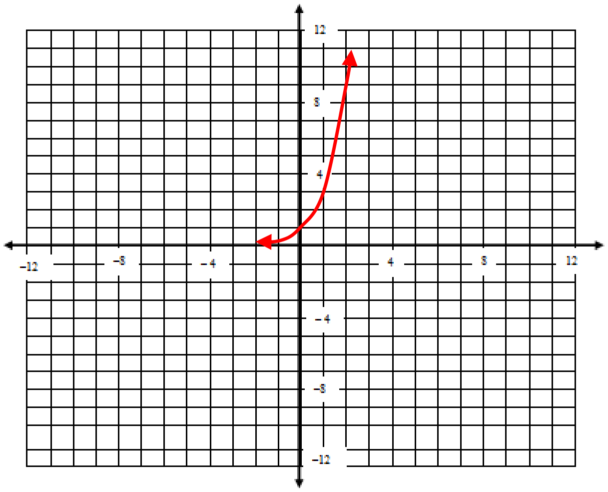
19. According to the function, the maximum profit is $15,988,000, which occurs when 20,000 security systems are installed.

21. According to the function, the maximum profit is $34,500, which occurs when 300 boots are produced and sold.

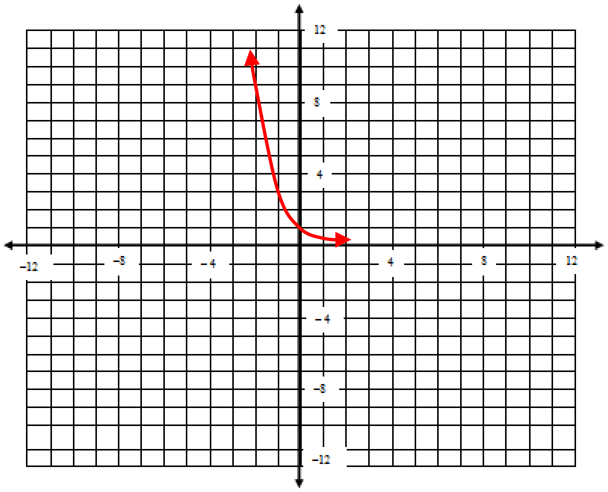
23. According to the function, the maximum height is 44.1 meters, which occurs 3 seconds after the object is launched. The time t must be more than 0 seconds because before that the object has not been launched. The time can be no more than 6 seconds because the object hits the ground 6 seconds after being launched.

Exercise Set 5.5

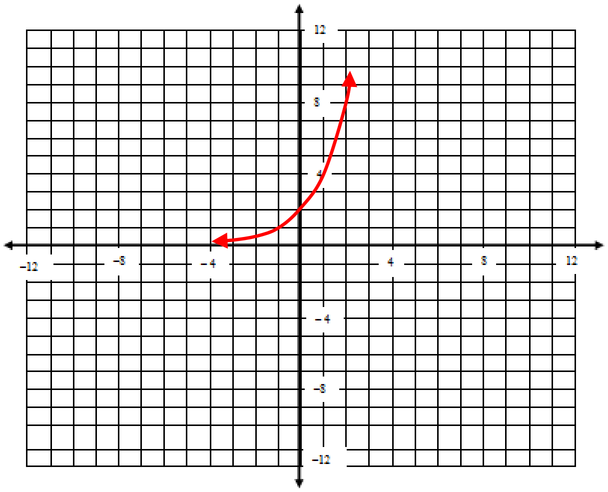
1. $f(x) = 3^x$



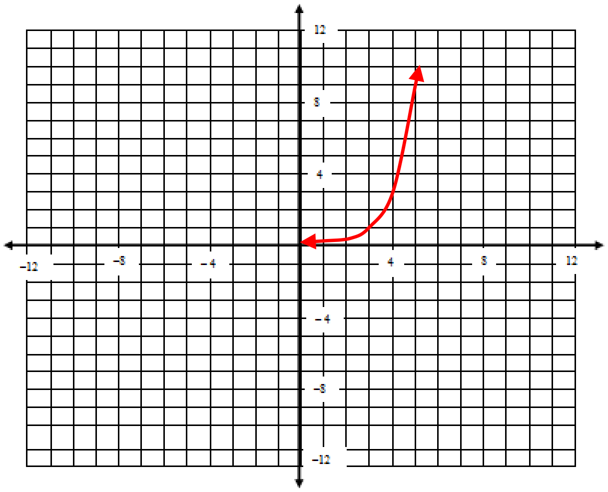
3. $f(x) = \left(\frac{1}{3}\right)^x$



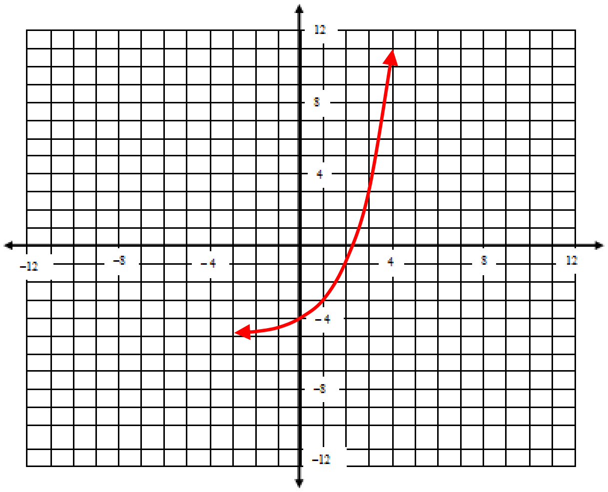
5. $f(x) = 2^{x+1}$



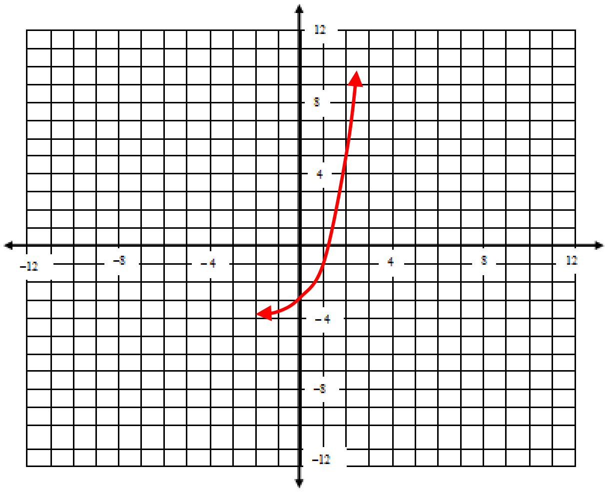
7. $f(x) = 3^{x-3}$



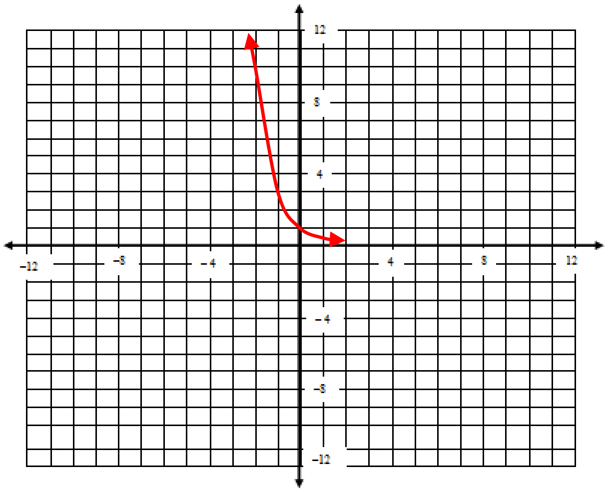
9. $f(x) = 2^{x} - 5$



11. $f(x) = 3^x - 4$



13. $f(x) = 3^{-x}$



15. According to the formula, 5,000,000 people had malaria in 1960 and 10,000,000 people had malaria in 1985. So, the doubling time is 25 years.

17. According to the formula, there will be 14.9 grams of carbon-14 10,000 years after the organism dies.

Exercise Set 5.6

1. $2 = \log_{10} 100$

3. $3 = \log_2 8$

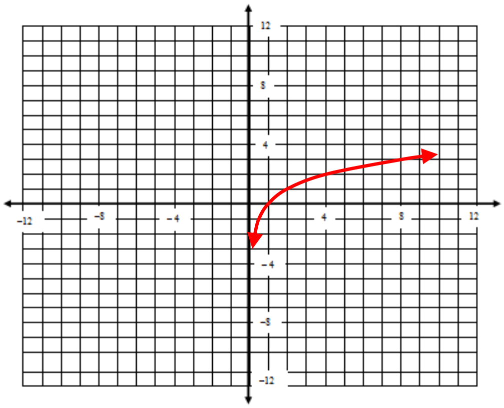
5. $-2 = \log_3 \frac{1}{9}$

7. $16 = 4^2$

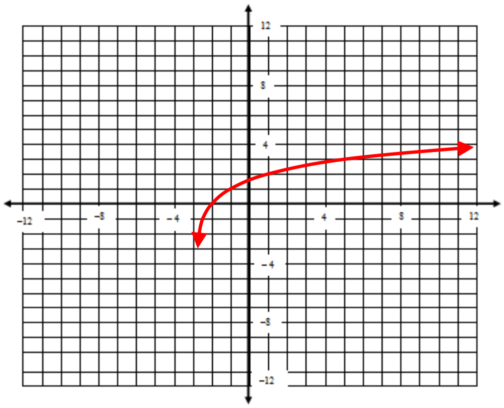
9. $1 = 5^0$

11. $\frac{1}{49} = 7^{-2}$

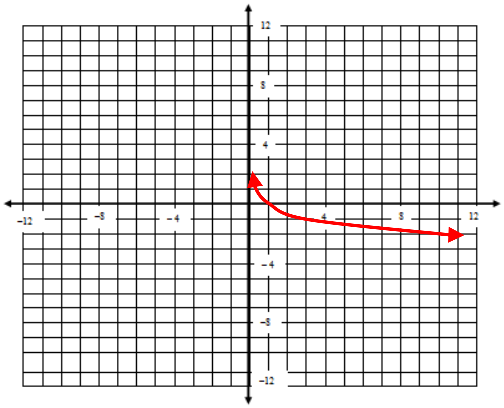
13. $y = \log_2 x$



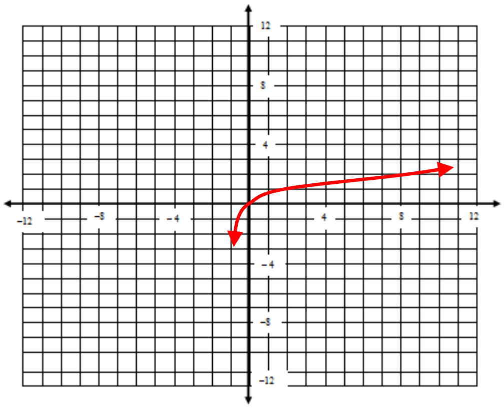
15. $y = \log_2(x + 3)$



17. $y = \log_{\frac{1}{3}} x$



19. $y = \log_3(x + 1)$



21. 4602 people will hear about the product after $40,000 is spent on advertising. 6176 people will hear about the product after $1,500,000 is spent on advertising. It does not appear that the extra advertising dollars will be well spent.

23. The growth rate is 9.9%, which is very high for inflation.

Exercise Set 6.1

1. $2080

3. $225,000

5. $162

7. $580.83

9. $184.93

11. $16,695.21

13. $25,920

15. $15,750

17. $14,978

19. 6.5%

21. 7.6%

23. $12,791

25. $8733

27. $69.35

29. $25,813.70

31. 3.7%

33. $48,781

Exercise Set 6.2

For future value problems, round to the nearest dollar.

1. $26,044

3. $557,265

5. $333,184

7. $25,871

9. $295,967

11. $207,484 ; If this were simple interest instead of annually compounded interest the future value would be $136,800. The future value for annually compounded interest is more than simple interest because the interest that is earned and applied each year grows for the rest of the investment time. For simple interest the interest is applied once at the end of the investment time based on the original principal only.

13. $125,225

15. $73,288

17. $1,076,891

19. $50,186

21. $8,723

23. $63,510

25. $63

For present value problems, round-up to the nearest dollar.

27. $24,263

29. $60,570

31. $35,801

33. $52,237

35. $11,655

37. $12,690

39. $349,460

41. a. $203,849
    b. $552,967
    c. $91,961 at 7% ; $452,469 at 3%
    d. The twenty-five year old could invest some money every month.

43. $39,445.74

Exercise Set 6.3

1. $158,853 ; $68,853

3. $1,489,153 ; $1,201,153

5. a. $603,518 ; $387,518
   b. $349,312 ; $133,312
   c. By starting to save for retirement earlier, the young person earns $254,206 extra in interest for the same amount of money put in.

7. $662 ; $721,960

9. $3900 ; $298,000

11. a. $558
    b. $1453
    c. It is important for the couple to start early because they pay less.
    d. $363 ; $752

13. $376,249 ; $229,249

15. $403,866 ; deposits account for $16,154,640; interest accounts for $3,845,360

Exercise Set 6.4

For the following questions, round to the nearest dollar.

1. a. $13,750
   b. $261,250
   c. $3918.75
   d. $1443
   e. $258,230

3. a. $174,000
   b. $406,000
   c. $8120
   d. $3054
   e. $143,720

5. a. $1802 ; $339,970
   b. $2615 ; $70,050
   c. If the couple can afford the down payment and the monthly payments, the second option will be much better because it saves them $269,920 in interest.

7. a. $1277 ; $387,960
   b. $1471 ; $84,780
   c. If the bachelor can afford the down payment and the monthly payments, the second option will be much better because it saves the bachelor $303,180 in interest.

9. $431 ; $17,720

11. $872 ; $1392

13. $776 ; $33,120

Exercise Set 7.1

1. 24

3. 1

5. 182

7. $639{,}200$

9. 28

11. $\approx 8.04531 \text{ x } 10^{13}$

13. $67{,}600{,}000$; Two possible license plates are AA 12345 and BZ 73912 .

15. 45 ; Two possible breakfast options are coffee/omelet/booth and juice/pancakes/counter .

17. ${}_{10}\mathrm{P}_2 = 90$

19. ${}_{20}\mathrm{P}_4 = 116{,}280$

21. ${}_{18}\mathrm{C}_4 = 3060$

23. ${}_{25}\mathrm{C}_6 = 177{,}100$

25. ${}_{10}\mathrm{C}_2 \cdot {}_{8}\mathrm{C}_5 = 2520$

27. 36

29. 32

31. ${}_{15}\mathrm{P}_4 = 32{,}760$

33. ${}_{30}\mathrm{P}_4 = 657{,}720$

35. ${}_{200}\mathrm{P}_3 = 7{,}880{,}400$

37. ${}_{150}\mathrm{C}_3 = 551{,}300$

39. ${}_{8}\mathrm{C}_3 \cdot {}_{15}\mathrm{C}_5 = 168{,}168$

41. 96

43. $20 \cdot 20 \cdot 20 = 8000$ ways if the flavors can be repeated. ${}_{20}\mathrm{P}_3 = 6840$ ways if the flavors must be different.

45. ${}_{100}\mathrm{C}_{15} \approx 2.53338 \text{ x } 10^{17}$

47. ${}_{80}\mathrm{P}_2 = 6320$

49. ${}_{15}\mathrm{C}_2 \cdot {}_{9}\mathrm{C}_3 = 8820$

Exercise Set 7.2

1. $\frac{3}{6}$ or .5

3. $\frac{2}{3}$ or .667

5. $\frac{1}{3}$ or .333

7. $\frac{0}{6}$ or 0

9. $\frac{3}{8}$ or .375

11. $\frac{1}{2}$ or .5

13. $\frac{1}{8}$ or .125

15. $\frac{7}{8}$ or .875

17. $\frac{0}{8}$ or 0

19. $\frac{3}{8}$ or .375

21. $\frac{11}{16}$ or .6875

23. $\frac{1}{4}$ or .25

25. $\frac{5}{16}$ or .3125

27. $\frac{11}{16}$ or .6875

29. $\frac{0}{16}$ or 0

31. $\frac{3}{10}$ or .3

33. $\frac{3}{10}$ or .3

35. $\frac{7}{15}$ or .467

37. $\frac{3}{5}$ or .6

39. $\frac{1}{5}$ or $.2$

41. $\frac{1}{2}$ or $.5$

43. $\frac{7}{12}$ or $.583$

45. $\frac{5}{18}$ or $.278$

47. $\frac{0}{36}$ or 0

49. $\frac{11}{20}$ or $.55$

51. $\frac{9}{20}$ or $.45$

53. $\frac{1}{6}$ or $.167$

55. $\frac{1}{5}$ or $.2$

57. $\frac{19}{50}$ or $.38$

59. $\frac{31}{100}$ or $.31$

61. $\frac{7}{50}$ or $.14$

Exercise Set 7.3

1. $.9999$ or 99.99%

3. $\frac{127}{128}$ or $.992$

5. $\frac{5}{8}$ or $.625$

7. $\frac{7}{16}$ or $.4375$

9. $\frac{7}{10}$ or $.7$

11. $\frac{1}{2}$ or $.5$

13. $\frac{2}{5}$ or $.4$

15. $\frac{1}{2}$ or $.5$

17. $\frac{3}{5}$ or $.6$

19. $\frac{1}{3}$ or $.333$

21. $\frac{3}{5}$ or $.6$

23. $\frac{2}{5}$ or $.4$

25. $\frac{1}{3}$ or $.333$

27. $\text{P(chocolate or walnut)} = \frac{18}{25}$ or $.72$ $\text{P(neither chocolate nor walnut)} = \frac{7}{25}$ or $.28$

29. $\frac{7}{15}$ or $.467$

31. $\frac{4}{5}$ or $.8$

33. $\frac{11}{20}$ or $.55$

35. $\frac{43}{60}$ or $.717$

37. $\frac{3}{4}$ or $.75$

39. $\frac{59}{100}$ or $.59$

41. $\frac{67}{100}$ or $.67$

43. $\frac{61}{100}$ or $.61$

45. $\frac{47}{100}$ or $.47$

47. $\frac{27}{50}$ or $.54$

49. $\frac{69}{100}$ or $.69$

Exercise Set 7.4

1. $\frac{1}{4}$ or $.25$

3. $\frac{1}{3}$ or $.333$

5. $\frac{1}{3}$ or $.333$

7. $\frac{1}{8}$ or $.125$

9. $\frac{1}{9}$ or $.111$

11. $\frac{8}{27}$ or $.296$

13. $\frac{4}{25}$ or $.16$

15. $\frac{2}{15}$ or $.133$

17. $\frac{9}{100}$ or $.09$

19. $\frac{1}{30}$ or $.0333$

21. $\frac{27}{1000}$ or $.027$

23. $\frac{1}{120}$ or $.00833$

25. $\frac{1}{4}$ or $.25$

27. $\frac{1}{3}$ or $.333$

29. a. 0.027
    b. 0.7
    c. 0.343
    d. 0.657

31. $\frac{49}{225}$ or $.217$

33. $\frac{1}{5}$ or $.2$

35. $\frac{1}{25}$ or $.04$

37. $\frac{1}{70}$ or $.0143$

39. $\frac{343}{3375}$ or $.102$

41. $\frac{1}{13}$ or $.0769$

43. $\frac{1}{7}$ or $.143$

45. $\frac{1}{5}$ or $.2$

47. $\frac{3}{7}$ or $.429$

49. $\frac{1}{5}$ or $.2$

51. $\frac{4}{9}$ or $.444$

53. $\frac{1}{6}$ or $.167$

55. $\frac{11}{20}$ or $.55$

57. $\frac{3}{8}$ or $.375$

59. $\frac{21}{100}$ or $.21$

61. $\frac{1}{10}$ or $.1$

63. $\frac{10}{21}$ or $.476$

65. $\frac{5}{21}$ or $.238$

67. $\frac{13}{20}$ or $.65$

69. $\frac{7}{18}$ or $.389$

Exercise Set 8.1

1. quantitative discrete

3. quantitative continuous

5. qualitative

7.

Number of Devices	Number of families
1	1
2	3
3	6
4	5
5	7
6	5
7	0
8	3

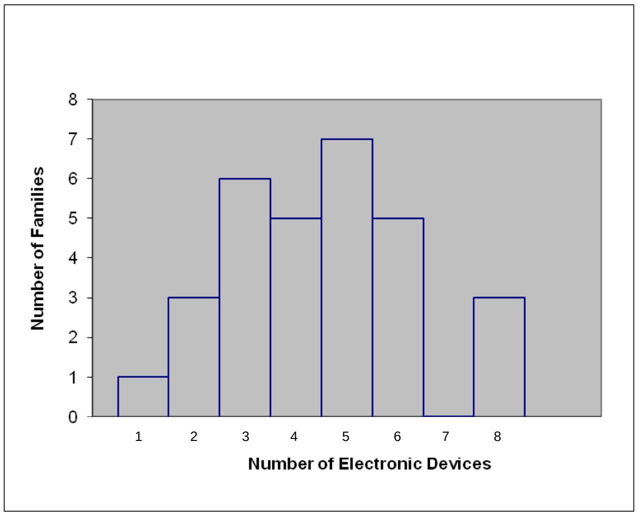
26 families had at least 3 electronic devices in their home.

9.

Number of Sunglasses	Number of Students
0	8
1	7
2	3
3	2

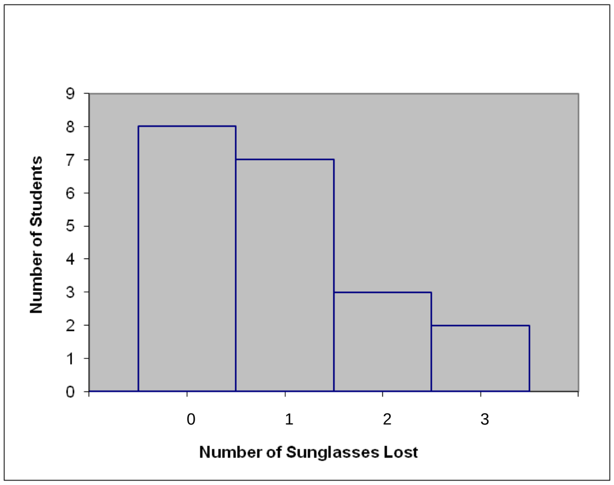
8 students did not lose any sunglasses over the last year.

11.



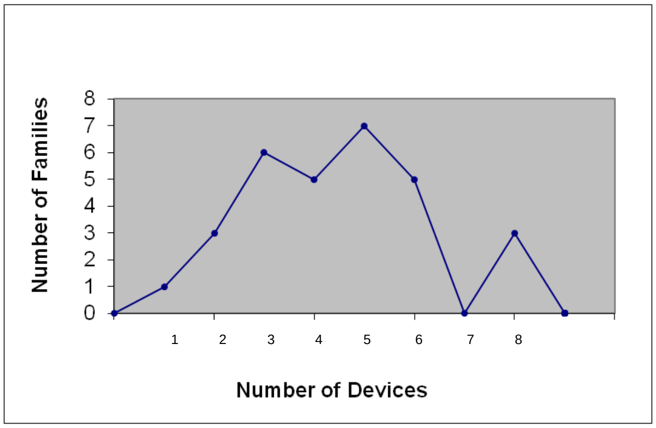
10 families had at most 3 electronic devices in their home.

13.



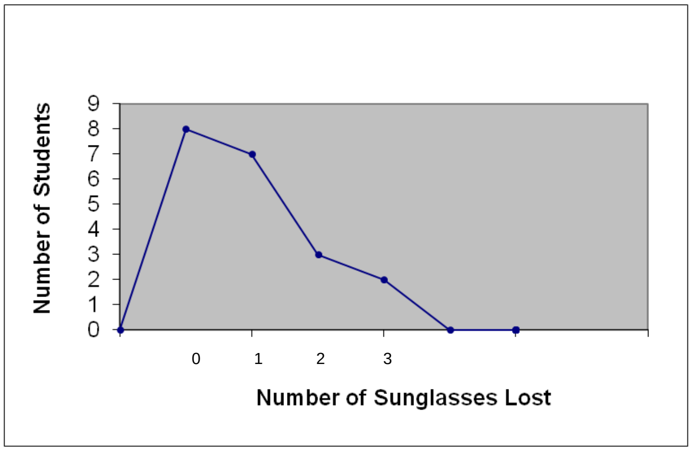
12 students lost at least 1 pair of sunglasses over the last year?

15.



14 families had between 2 and 4 electronic devices inclusive in their home.

17.



All 20 students lost less than 4 pairs of sunglasses over the last year.

19. Frequency distributions, frequency polygons, and histograms all group the raw data into different classes and display the frequency of the different classes. Frequency polygons and histograms display the data as pictures whereas frequency polygons use a table with numbers to show the size of the classes. Histograms use the height of bars to represent the size of the different classes.

Exercise Set 8.2

1. 87 is the mean grade.

3. 90 is the median grade.

5. A score of 80 is the mean grade.

7. A score of 75 is the median grade.

9. Answers will vary. The mean takes the size of all the data values into account in its calculation. The mode is the most frequently occurring value. The median is the “middle” value for all of the values. The midrange is the “middle” value for the lowest and highest values only.

11. 0 absences

13. 3 absences

15. 4.5 is the mode of the scores.

17. 4.15 is the midrange of the scores.

19. 4.4 children

21. 4.5 children

23. 11.4 cartoons of eggs

25. 11.5 cartoons of eggs

27. 2.2 cups of coffee

29. 2 cups of coffee

31. Answers will vary. The mean takes the size of all the data values into account in its calculation. The mode is the most frequently occurring value. The median is the “middle” value for all of the values. The midrange is the “middle” value for the lowest and highest values only.

33. 0 lost pairs of sunglasses

35. 1.5 lost pairs of sunglasses

37. 4 false claims

39. 4.5 false claims

41. 6.4 lost keys

43. 6 lost keys

Exercise Set 8.3

1. The range for the grades is 30.

3. The range for the grades is 35.

5. Set 1: The mean score is 80.
   Set 2: The mean score is 80.

7. Set 1: The standard deviation of the test scores is 20.
   Set 2: The standard deviation of the test scores is 14.6.

9. The standard deviations show that the data for set 1 is more spread out than the data for set 2 even though the means and ranges are the same..

11. 2.3 absences

13. The standard deviation of the scores is 0.44 .

15. 1.8 lost children

17. 16 eggs

19. 1.5 cups of coffee

21. 0.999 lost pairs of sunglasses

23. 1.75 false claims

25. 1.5 lost keys

Exercise Set 8.4

1. It means that 95% of men are less than 74 inches tall.

3. It means that 92% of women are less than 69 inches tall.

5. It means that 98% of the scores on the test are less than 52.

7. 95%

9. 16%

11. 2.5%

13. 68%

15. 2.5%

17. 16%

19. 2

21. -1.5

23. 1.83

25. 0

27. -0.33

29. 0.9821 or 98.21%

31. 0.0901 or 9.01%

33. 0.3936 or 39.36%

35. 0.9838 or 98.38%

37. 0.6141 or 61.41%

39. 0.8943 or 89.43%

41. 0.5346 or 53.46%

43. 0.9452 or 94.52%

45. 0.9641 or 96.41%

47. 0.0294 or 2.94%

49. 0.2877 or 28.77%

51. 0.4973 or 49.73%

53. 0.4246 or 42.46%

55. 0.903 or 90.3%

57. 0.1151 or 11.51%

59. 0.9452 or 94.52%

61. 0.6962 or 69.62%

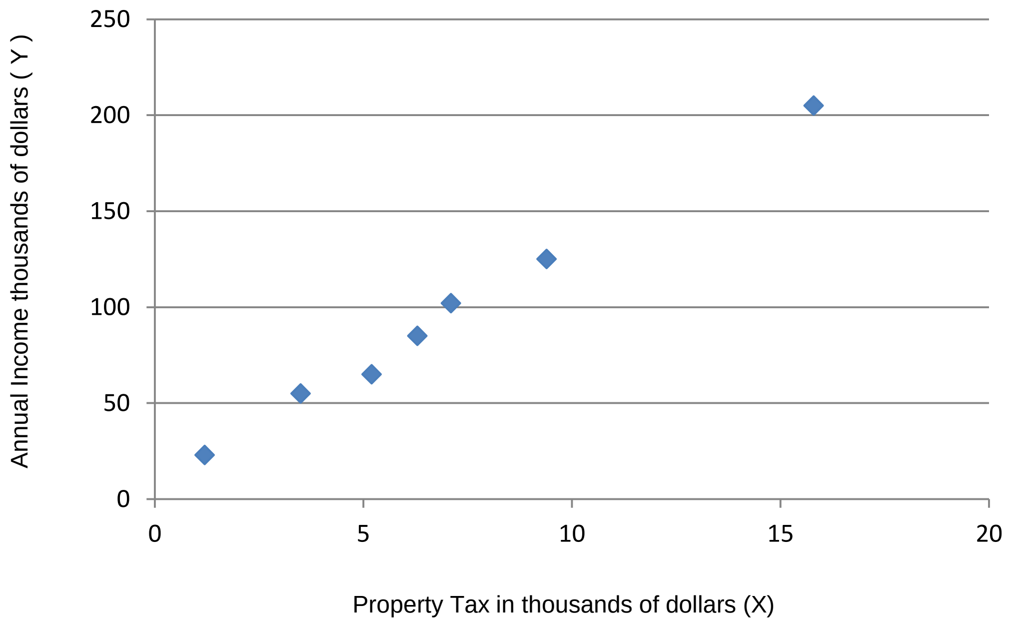
63. 0.0012 or 0.12%

65. 0.0012 or 0.12%

67. 0.3167 or 31.67%

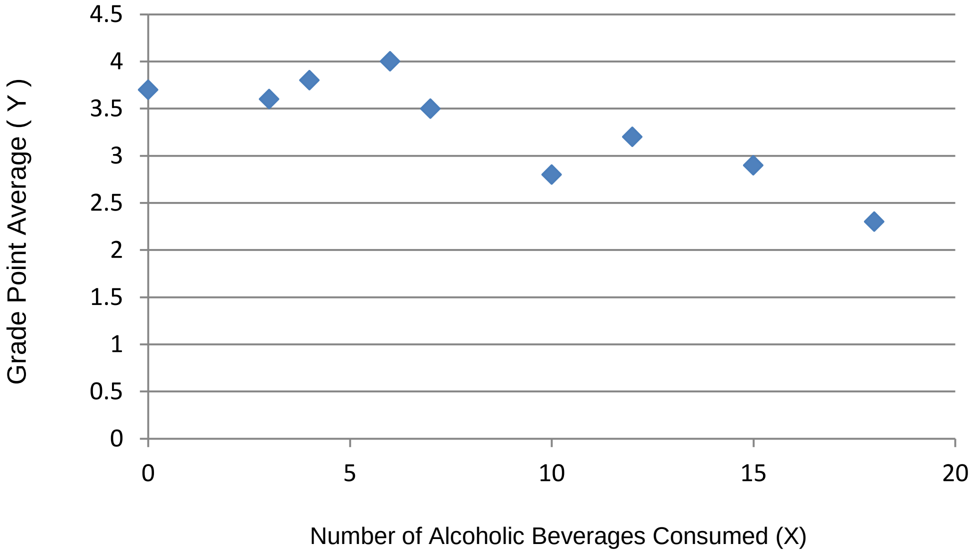
Exercise Set 8.5

1.



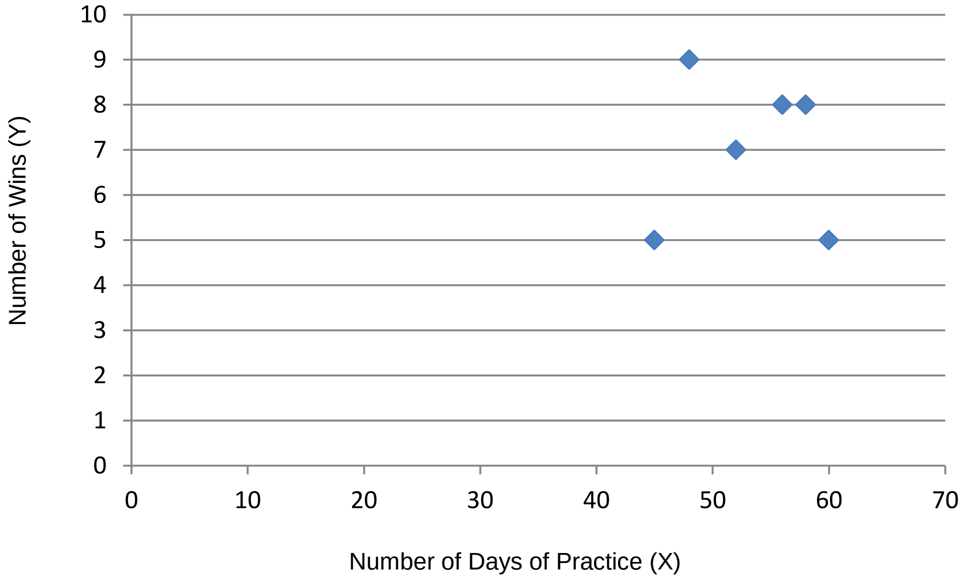
There is a positive linear correlation between property tax and annual income.

3.



There is a negative linear correlation between number of alcoholic beverages consumed and grade point average.

5.



There is no linear correlation between number of days of practice and number of wins.

7. D

9. E

11. A

13. r = -0.96 ; strong negative correlation ; supported by the scatter plot

15. r = 0.69 ; moderate positive linear correlation ; supported by the scatter plot

17. r = 0.915 ; strong positive linear correlation ; support by the scatter plot

Math 103 Formula Sheet

Financial Management

Simple Interest: $\text{Int} = \text{Prt}$

Future Value for Simple Interest: $\text{FV} = \text{P}(1 + \text{rt})$

Future Value for Compound Interest: $\text{FV} = \text{P}\left(1 + \frac{\text{r}}{\text{n}}\right)^{\text{nt}}$

Present Value for Compound Interest: $\text{P} = \dfrac{\text{FV}}{\left(1 + \frac{\text{r}}{\text{n}}\right)^{\text{nt}}}$

Future Value for continuous compounding: $\text{FV} = \text{Pe}^{\text{r}\cdot\text{t}}$

Effective Annual Yield: $\text{EAY} = \left(1 + \frac{\text{r}}{\text{n}}\right)^{\text{n}} - 1$

Future Value of an Annuity (Pmt is the amount of each deposit): $\text{FV} = \dfrac{\text{Pmt}\left[\left(1 + \frac{\text{r}}{\text{n}}\right)^{\text{nt}} - 1\right]}{\left(\frac{\text{r}}{\text{n}}\right)}$

Periodic deposits for an Annuity (FV is the future value of the annuity): $\text{Pmt} = \dfrac{\text{FV}\left(\frac{\text{r}}{\text{n}}\right)}{\left[\left(1 + \frac{\text{r}}{\text{n}}\right)^{\text{nt}} - 1\right]}$

Periodic Mortgage Payments (B is the amount of mortgage): $\text{Pmt} = \dfrac{\text{B}\left(\frac{\text{r}}{\text{n}}\right)}{\left[1 - \left(1 + \frac{\text{r}}{\text{n}}\right)^{-\text{nt}}\right]}$

Probability and Counting Rules

Permutation rule: ${}_{\text{n}}\text{P}_{\text{k}} = \dfrac{\text{n}!}{(\text{n} - \text{k})!}$

Combination rule: ${}_{\text{n}}\text{C}_{\text{k}} = \dfrac{\text{n}!}{(\text{n} - \text{k})!\,\text{k}!}$

$\text{P}(\overline{\text{E}}) = 1 - \text{P}(\text{E})$

$\text{P}(\text{E}) = 1 - \text{P}(\overline{\text{E}})$

$\text{P}(\text{A or B}) = \text{P}(\text{A}) + \text{P}(\text{B})$

$\text{P}(\text{A or B}) = \text{P}(\text{A}) + \text{P}(\text{B}) - \text{P}(\text{A and B})$

$\text{P}(\text{A and B}) = \text{P}(\text{A}) \cdot \text{P}(\text{B})$

$\text{P}(\text{B given A}) = \dfrac{\text{number of common outcomes for B and A}}{\text{number of outcomes within A}}$

Statistics

Mean for individual data: $\bar{\text{x}} = \dfrac{\sum \text{x}}{\text{n}}$

Mean for grouped data: $\bar{\text{x}} = \dfrac{\sum \text{f} \cdot \text{x}_{\text{m}}}{\text{n}}$

Standard Deviation: $\text{s} = \sqrt{\dfrac{\sum(\text{x} - \bar{\text{x}})^2}{\text{n} - 1}}$

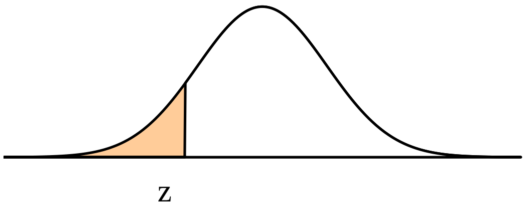
Z-score: $\text{z} = \dfrac{\text{x} - \bar{\text{x}}}{\text{s}}$

$\bar{\text{x}}$ = mean x= data values $\sum$ = add all the values f = frequency x_{m}= class or class midpoint s = standard deviation

Standard Normal Distribution Cumulative Probabilities (Percentiles)

Table Values represent area to the left of z

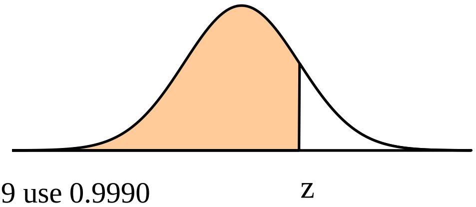
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.00	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.90	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.80	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.70	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.60	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.50	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.40	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.30	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.20	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.10	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.00	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.90	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.80	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.70	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.60	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.50	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.40	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.30	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.20	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.10	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.00	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.90	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.80	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.70	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.60	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.50	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.40	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.30	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.20	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.10	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.00	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641



For values of z less than -3.09 use 0.0010

Standard Normal Distribution Cumulative Probabilities (Percentiles)
Table Values represent area to the left of z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990



For values of z more than 3.09 use 0.9990